Chapter

# THE MULTIVARIATE NORMAL DISTRIBUTION

## 4.1 Introduction

A generalization of the familiar bell-shaped normal density to several dimensions plays a fundamental role in multivariate analysis. In fact, most of the techniques encountered in this book are based on the assumption that the data were generated from a *multivariate* normal distribution. While real data are never *exactly* multivariate normal, the normal density is often a useful approximation to the "true" population distribution.

One advantage of the multivariate normal distribution stems from the fact that it is mathematically tractable and "nice" results can be obtained. This is frequently not the case for other data-generating distributions. Of course, mathematical attractiveness per se is of little use to the practitioner. It turns out, however, that normal distributions are useful in practice for two reasons: First, the normal distribution serves as a bona fide population model in some instances; second, the sampling distributions of many multivariate statistics are approximately normal, regardless of the form of the parent population, because of a *central limit* effect.

To summarize, many real-world problems fall naturally within the framework of normal theory. The importance of the normal distribution rests on its dual role as both population model for certain natural phenomena and approximate sampling distribution for many statistics.

## 4.2 The Multivariate Normal Density and Its Properties

The multivariate normal density is a generalization of the univariate normal density to $p \geq 2$ dimensions. Recall that the univariate normal distribution, with mean $\mu$ and variance $\sigma^2$, has the probability density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-[(x-\mu)/\sigma]^2/2} \qquad -\infty < x < \infty \tag{4-1}$$

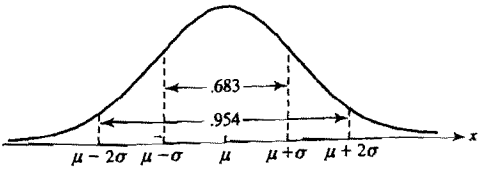


**Figure 4.1** A normal density with mean $\mu$ and variance $\sigma^2$ and selected areas under the curve.

A plot of this function yields the familiar bell-shaped curve shown in Figure 4.1. Also shown in the figure are approximate areas under the curve within $\pm 1$ standard deviations and $\pm 2$ standard deviations of the mean. These areas represent probabilities, and thus, for the normal random variable $X$,

$$P(\mu - \sigma \leq X \leq \mu + \sigma) \doteq .68$$

$$P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \doteq .95$$

It is convenient to denote the normal density function with mean $\mu$ and variance $\sigma^2$ by $N(\mu, \sigma^2)$. Therefore, $N(10, 4)$ refers to the function in (4-1) with $\mu = 10$ and $\sigma = 2$. This notation will be extended to the multivariate case later.

The term

$$\left(\frac{x - \mu}{\sigma}\right)^2 = (x - \mu)(\sigma^2)^{-1}(x - \mu) \tag{4-2}$$

in the exponent of the univariate normal density function measures the square of the distance from $x$ to $\mu$ in standard deviation units. This can be generalized for a $p \times 1$ vector $\mathbf{x}$ of observations on several variables as

$$(\mathbf{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}) \tag{4-3}$$

The $p \times 1$ vector $\boldsymbol{\mu}$ represents the expected value of the random vector $\mathbf{X}$, and the $p \times p$ matrix $\boldsymbol{\Sigma}$ is the variance–covariance matrix of $\mathbf{X}$. [See (2–30) and (2–31).] We shall assume that the symmetric matrix $\boldsymbol{\Sigma}$ is positive definite, so the expression in (4-3) is the square of the generalized distance from $\mathbf{x}$ to $\boldsymbol{\mu}$.

The multivariate normal density is obtained by replacing the univariate distance in (4-2) by the multivariate generalized distance of (4-3) in the density function of (4-1). When this replacement is made, the univariate normalizing constant $(2\pi)^{-1/2}(\sigma^2)^{-1/2}$ must be changed to a more general constant that makes the *volume* under the surface of the multivariate density function unity for any $p$. This is necessary because, in the multivariate case, probabilities are represented by volumes under the surface over regions defined by intervals of the $x_i$ values. It can be shown (see [1]) that this constant is $(2\pi)^{-p/2}|\boldsymbol{\Sigma}|^{-1/2}$, and consequently, a $p$-dimensional normal density for the random vector $\mathbf{X}' = [X_1, X_2, \ldots, X_p]$ has the form

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{p/2}|\boldsymbol{\Sigma}|^{1/2}} e^{-(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})/2} \tag{4-4}$$

where $-\infty < x_i < \infty$, $i = 1, 2, \ldots, p$. We shall denote this $p$-dimensional normal density by $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, which is analogous to the normal density in the univariate case.

**Example 4.1 (Bivariate normal density)** Let us evaluate the $p = 2$-variate normal density in terms of the individual parameters $\mu_1 = E(X_1)$, $\mu_2 = E(X_2)$, $\sigma_{11} = \text{Var}(X_1)$, $\sigma_{22} = \text{Var}(X_2)$, and $\rho_{12} = \sigma_{12}/(\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}) = \text{Corr}(X_1, X_2)$.

Using Result 2A.8, we find that the inverse of the covariance matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$$

is

$$\boldsymbol{\Sigma}^{-1} = \frac{1}{\sigma_{11}\sigma_{22} - \sigma_{12}^2} \begin{bmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{bmatrix}$$

Introducing the correlation coefficient $\rho_{12}$ by writing $\sigma_{12} = \rho_{12}\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}$, we obtain $\sigma_{11}\sigma_{22} - \sigma_{12}^2 = \sigma_{11}\sigma_{22}(1 - \rho_{12}^2)$, and the squared distance becomes

$$\begin{aligned}
&(\mathbf{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}) \\
&= [x_1 - \mu_1, x_2 - \mu_2] \frac{1}{\sigma_{11}\sigma_{22}(1 - \rho_{12}^2)} \\
&\qquad \begin{bmatrix} \sigma_{22} & -\rho_{12}\sqrt{\sigma_{11}}\sqrt{\sigma_{22}} \\ -\rho_{12}\sqrt{\sigma_{11}}\sqrt{\sigma_{22}} & \sigma_{11} \end{bmatrix} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix} \\
&= \frac{\sigma_{22}(x_1 - \mu_1)^2 + \sigma_{11}(x_2 - \mu_2)^2 - 2\rho_{12}\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}(x_1 - \mu_1)(x_2 - \mu_2)}{\sigma_{11}\sigma_{22}(1 - \rho_{12}^2)} \\
&= \frac{1}{1 - \rho_{12}^2}\left[\left(\frac{x_1 - \mu_1}{\sqrt{\sigma_{11}}}\right)^2 + \left(\frac{x_2 - \mu_2}{\sqrt{\sigma_{22}}}\right)^2 - 2\rho_{12}\left(\frac{x_1 - \mu_1}{\sqrt{\sigma_{11}}}\right)\left(\frac{x_2 - \mu_2}{\sqrt{\sigma_{22}}}\right)\right] \qquad (4\text{-}5)
\end{aligned}$$

The last expression is written in terms of the standardized values $(x_1 - \mu_1)/\sqrt{\sigma_{11}}$ and $(x_2 - \mu_2)/\sqrt{\sigma_{22}}$.

Next, since $|\boldsymbol{\Sigma}| = \sigma_{11}\sigma_{22} - \sigma_{12}^2 = \sigma_{11}\sigma_{22}(1 - \rho_{12}^2)$, we can substitute for $\boldsymbol{\Sigma}^{-1}$ and $|\boldsymbol{\Sigma}|$ in (4-4) to get the expression for the bivariate ($p = 2$) normal density involving the individual parameters $\mu_1, \mu_2, \sigma_{11}, \sigma_{22}$, and $\rho_{12}$:

$$\begin{aligned}
f(x_1, x_2) = &\frac{1}{2\pi\sqrt{\sigma_{11}\sigma_{22}(1 - \rho_{12}^2)}} \qquad (4\text{-}6) \\
&\times \exp\left\{-\frac{1}{2(1 - \rho_{12}^2)}\left[\left(\frac{x_1 - \mu_1}{\sqrt{\sigma_{11}}}\right)^2 + \left(\frac{x_2 - \mu_2}{\sqrt{\sigma_{22}}}\right)^2 \right.\right. \\
&\qquad \left.\left. - 2\rho_{12}\left(\frac{x_1 - \mu_1}{\sqrt{\sigma_{11}}}\right)\left(\frac{x_2 - \mu_2}{\sqrt{\sigma_{22}}}\right)\right]\right\}
\end{aligned}$$

The expression in (4-6) is somewhat unwieldy, and the compact general form in (4-4) is more informative in many ways. On the other hand, the expression in (4-6) is useful for discussing certain properties of the normal distribution. For example, if the random variables $X_1$ and $X_2$ are uncorrelated, so that $\rho_{12} = 0$, the joint density can be written as the product of two univariate normal densities each of the form of (4-1).

That is, $f(x_1, x_2) = f(x_1)f(x_2)$ and $X_1$ and $X_2$ are independent. [See (2-28).] This result is true in general. (See Result 4.5.)

Two bivariate distributions with $\sigma_{11} = \sigma_{22}$ are shown in Figure 4.2. In Figure 4.2(a), $X_1$ and $X_2$ are independent ($\rho_{12} = 0$). In Figure 4.2(b), $\rho_{12} = .75$. Notice how the presence of correlation causes the probability to concentrate along a line. ■

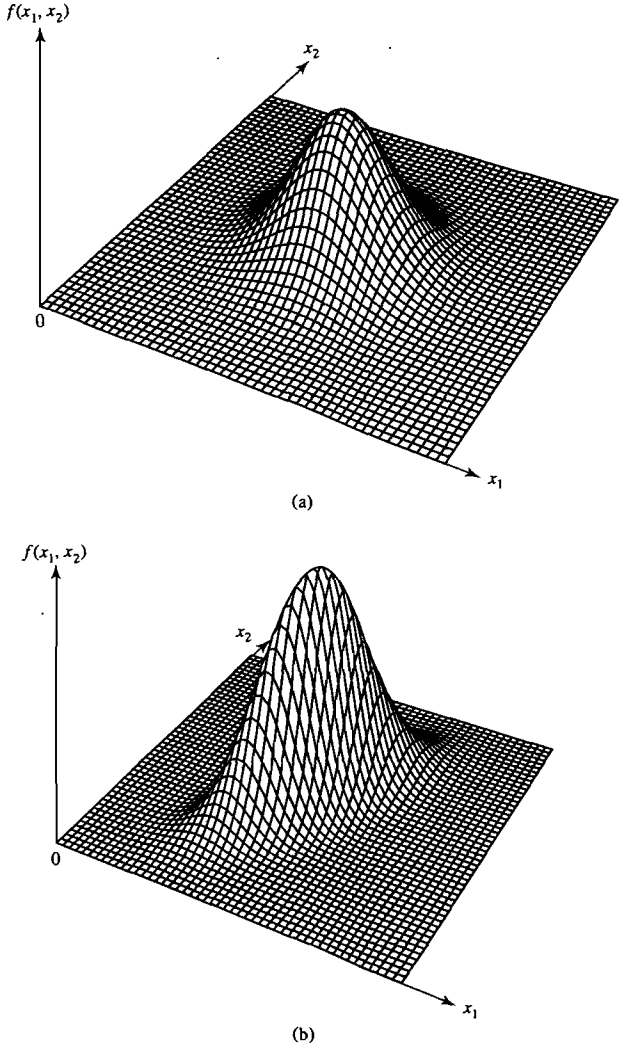


**Figure 4.2** Two bivariate normal distributions. (a) $\sigma_{11} = \sigma_{22}$ and $\rho_{12} = 0$. (b) $\sigma_{11} = \sigma_{22}$ and $\rho_{12} = .75$.

From the expression in (4-4) for the density of a $p$-dimensional normal variable, it should be clear that the paths of $\mathbf{x}$ values yielding a constant height for the density are ellipsoids. That is, the multivariate normal density is constant on surfaces where the square of the distance $(\mathbf{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})$ is constant. These paths are called *contours*:

$$\begin{aligned}\textit{Constant probability density contour} &= \{\text{all } \mathbf{x} \text{ such that } (\mathbf{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}) = c^2\} \\ &= \text{surface of an ellipsoid centered at } \boldsymbol{\mu}\end{aligned}$$

The axes of each ellipsoid of constant density are in the direction of the eigenvectors of $\boldsymbol{\Sigma}^{-1}$, and their lengths are proportional to the reciprocals of the square roots of the eigenvalues of $\boldsymbol{\Sigma}^{-1}$. Fortunately, we can avoid the calculation of $\boldsymbol{\Sigma}^{-1}$ when determining the axes, since these ellipsoids are also determined by the eigenvalues and eigenvectors of $\boldsymbol{\Sigma}$. We state the correspondence formally for later reference.

**Result 4.1.** If $\boldsymbol{\Sigma}$ is positive definite, so that $\boldsymbol{\Sigma}^{-1}$ exists, then

$$\boldsymbol{\Sigma}\mathbf{e} = \lambda\mathbf{e} \quad \text{implies} \quad \boldsymbol{\Sigma}^{-1}\mathbf{e} = \left(\frac{1}{\lambda}\right)\mathbf{e}$$

so $(\lambda, \mathbf{e})$ is an eigenvalue–eigenvector pair for $\boldsymbol{\Sigma}$ corresponding to the pair $(1/\lambda, \mathbf{e})$ for $\boldsymbol{\Sigma}^{-1}$. Also, $\boldsymbol{\Sigma}^{-1}$ is positive definite.

**Proof.** For $\boldsymbol{\Sigma}$ positive definite and $\mathbf{e} \neq \mathbf{0}$ an eigenvector, we have $0 < \mathbf{e}'\boldsymbol{\Sigma}\mathbf{e} = \mathbf{e}'(\boldsymbol{\Sigma}\mathbf{e}) = \mathbf{e}'(\lambda\mathbf{e}) = \lambda\mathbf{e}'\mathbf{e} = \lambda$. Moreover, $\mathbf{e} = \boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}\mathbf{e}) = \boldsymbol{\Sigma}^{-1}(\lambda\mathbf{e})$, or $\mathbf{e} = \lambda\boldsymbol{\Sigma}^{-1}\mathbf{e}$, and division by $\lambda > 0$ gives $\boldsymbol{\Sigma}^{-1}\mathbf{e} = (1/\lambda)\mathbf{e}$. Thus, $(1/\lambda, \mathbf{e})$ is an eigenvalue–eigenvector pair for $\boldsymbol{\Sigma}^{-1}$. Also, for any $p \times 1$ $\mathbf{x}$, by (2-21)

$$\begin{aligned}\mathbf{x}'\boldsymbol{\Sigma}^{-1}\mathbf{x} &= \mathbf{x}'\left(\sum_{i=1}^{p}\left(\frac{1}{\lambda_i}\right)\mathbf{e}_i\mathbf{e}_i'\right)\mathbf{x} \\ &= \sum_{i=1}^{p}\left(\frac{1}{\lambda_i}\right)(\mathbf{x}'\mathbf{e}_i)^2 \geq 0\end{aligned}$$

since each term $\lambda_i^{-1}(\mathbf{x}'\mathbf{e}_i)^2$ is nonnegative. In addition, $\mathbf{x}'\mathbf{e}_i = 0$ for all $i$ only if $\mathbf{x} = \mathbf{0}$. So $\mathbf{x} \neq \mathbf{0}$ implies that $\sum_{i=1}^{p}(1/\lambda_i)(\mathbf{x}'\mathbf{e}_i)^2 > 0$, and it follows that $\boldsymbol{\Sigma}^{-1}$ is positive definite. ■

The following summarizes these concepts:

Contours of constant density for the $p$-dimensional normal distribution are ellipsoids defined by $\mathbf{x}$ such the that

$$(\mathbf{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}) = c^2 \tag{4-7}$$

These ellipsoids are centered at $\boldsymbol{\mu}$ and have axes $\pm c\sqrt{\lambda_i}\,\mathbf{e}_i$, where $\boldsymbol{\Sigma}\mathbf{e}_i = \lambda_i\mathbf{e}_i$ for $i = 1, 2, \ldots, p$.

A contour of constant density for a bivariate normal distribution with $\sigma_{11} = \sigma_{22}$ is obtained in the following example.

**Example 4.2 (Contours of the bivariate normal density)** We shall obtain the axes of constant probability density contours for a bivariate normal distribution when $\sigma_{11} = \sigma_{22}$. From (4-7), these axes are given by the eigenvalues and eigenvectors of $\Sigma$. Here $|\Sigma - \lambda \mathbf{I}| = 0$ becomes

$$0 = \begin{vmatrix} \sigma_{11} - \lambda & \sigma_{12} \\ \sigma_{12} & \sigma_{11} - \lambda \end{vmatrix} = (\sigma_{11} - \lambda)^2 - \sigma_{12}^2$$

$$= (\lambda - \sigma_{11} - \sigma_{12})(\lambda - \sigma_{11} + \sigma_{12})$$

Consequently, the eigenvalues are $\lambda_1 = \sigma_{11} + \sigma_{12}$ and $\lambda_2 = \sigma_{11} - \sigma_{12}$. The eigenvector $\mathbf{e}_1$ is determined from

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{11} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = (\sigma_{11} + \sigma_{12}) \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

or

$$\sigma_{11}e_1 + \sigma_{12}e_2 = (\sigma_{11} + \sigma_{12})e_1$$
$$\sigma_{12}e_1 + \sigma_{11}e_2 = (\sigma_{11} + \sigma_{12})e_2$$

These equations imply that $e_1 = e_2$, and after normalization, the first eigenvalue-eigenvector pair is

$$\lambda_1 = \sigma_{11} + \sigma_{12}, \qquad \mathbf{e}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Similarly, $\lambda_2 = \sigma_{11} - \sigma_{12}$ yields the eigenvector $\mathbf{e}_2' = [1/\sqrt{2}, -1/\sqrt{2}]$.

When the covariance $\sigma_{12}$ (or correlation $\rho_{12}$) is positive, $\lambda_1 = \sigma_{11} + \sigma_{12}$ is the *largest* eigenvalue, and its associated eigenvector $\mathbf{e}_1' = [1/\sqrt{2}, 1/\sqrt{2}]$ lies along the 45° line through the point $\boldsymbol{\mu}' = [\mu_1, \mu_2]$. This is true for any positive value of the covariance (correlation). Since the axes of the constant-density ellipses are given by $\pm c\sqrt{\lambda_1}\,\mathbf{e}_1$ and $\pm c\sqrt{\lambda_2}\,\mathbf{e}_2$ [see (4-7)], and the eigenvectors each have length unity, the major axis will be associated with the largest eigenvalue. For positively correlated normal random variables, then, the *major* axis of the constant-density ellipses will be along the 45° line through $\boldsymbol{\mu}$. (See Figure 4.3.)

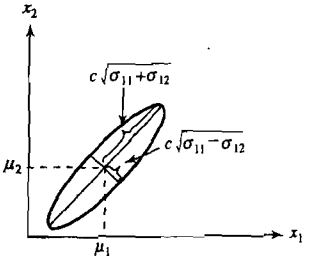


**Figure 4.3** A constant-density contour for a bivariate normal distribution with $\sigma_{11} = \sigma_{22}$ and $\sigma_{12} > 0$ (or $\rho_{12} > 0$).

When the covariance (correlation) is negative, $\lambda_2 = \sigma_{11} - \sigma_{12}$ will be the largest eigenvalue, and the major axes of the constant-density ellipses will lie along a line at right angles to the 45° line through $\boldsymbol{\mu}$. (These results are true only for $\sigma_{11} = \sigma_{22}$.)

To summarize, the axes of the ellipses of constant density for a bivariate normal distribution with $\sigma_{11} = \sigma_{22}$ are determined by

$$\pm c\sqrt{\sigma_{11} + \sigma_{12}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \text{and} \quad \pm c\sqrt{\sigma_{11} - \sigma_{12}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

■

We show in Result 4.7 that the choice $c^2 = \chi^2_p(\alpha)$, where $\chi^2_p(\alpha)$ is the upper $(100\alpha)$th percentile of a chi-square distribution with $p$ degrees of freedom, leads to contours that contain $(1 - \alpha) \times 100\%$ of the probability. Specifically, the following is true for a $p$-dimensional normal distribution:

The solid ellipsoid of $\mathbf{x}$ values satisfying

$$(\mathbf{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}) \leq \chi^2_p(\alpha) \tag{4-8}$$

has probability $1 - \alpha$.

The constant-density contours containing 50% and 90% of the probability under the bivariate normal surfaces in Figure 4.2 are pictured in Figure 4.4.

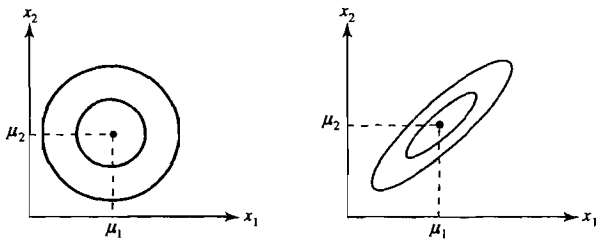


**Figure 4.4** The 50% and 90% contours for the bivariate normal distributions in Figure 4.2.

The $p$-variate normal density in (4-4) has a maximum value when the squared distance in (4-3) is zero—that is, when $\mathbf{x} = \boldsymbol{\mu}$. Thus, $\boldsymbol{\mu}$ is the point of maximum density, or *mode*, as well as the expected value of $\mathbf{X}$, or *mean*. The fact that $\boldsymbol{\mu}$ is the mean of the multivariate normal distribution follows from the symmetry exhibited by the constant-density contours: These contours are centered, or balanced, at $\boldsymbol{\mu}$.

## Additional Properties of the Multivariate Normal Distribution

Certain properties of the normal distribution will be needed repeatedly in our explanations of statistical models and methods. These properties make it possible to manipulate normal distributions easily and, as we suggested in Section 4.1, are partly responsible for the popularity of the normal distribution. The key properties, which we shall soon discuss in some mathematical detail, can be stated rather simply.

The following are true for a random vector $\mathbf{X}$ having a multivariate normal distribution:

1. Linear combinations of the components of $\mathbf{X}$ are normally distributed.
2. All subsets of the components of $\mathbf{X}$ have a (multivariate) normal distribution.
3. Zero covariance implies that the corresponding components are independently distributed.
4. The conditional distributions of the components are (multivariate) normal.

These statements are reproduced mathematically in the results that follow. Many of these results are illustrated with examples. The proofs that are included should help improve your understanding of matrix manipulations and also lead you to an appreciation for the manner in which the results successively build on themselves.

Result 4.2 can be taken as a working definition of the normal distribution. With this in hand, the subsequent properties are almost immediate. Our partial proof of Result 4.2 indicates how the linear combination definition of a normal density relates to the multivariate density in (4-4).

**Result 4.2.** If $\mathbf{X}$ is distributed as $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, then any linear combination of variables $\mathbf{a}'\mathbf{X} = a_1X_1 + a_2X_2 + \cdots + a_pX_p$ is distributed as $N(\mathbf{a}'\boldsymbol{\mu}, \mathbf{a}'\boldsymbol{\Sigma}\mathbf{a})$. Also, if $\mathbf{a}'\mathbf{X}$ is distributed as $N(\mathbf{a}'\boldsymbol{\mu}, \mathbf{a}'\boldsymbol{\Sigma}\mathbf{a})$ for every $\mathbf{a}$, then $\mathbf{X}$ must be $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

**Proof.** The expected value and variance of $\mathbf{a}'\mathbf{X}$ follow from (2-43). Proving that $\mathbf{a}'\mathbf{X}$ is normally distributed if $\mathbf{X}$ is multivariate normal is more difficult. You can find a proof in [1]. The second part of result 4.2 is also demonstrated in [1]. ■

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**Example 4.3 (The distribution of a linear combination of the components of a normal random vector)** Consider the linear combination $\mathbf{a}'\mathbf{X}$ of a multivariate normal random vector determined by the choice $\mathbf{a}' = [1, 0, \ldots, 0]$. Since

$$\mathbf{a}'\mathbf{X} = [1, 0, \ldots, 0]\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix} = X_1$$

and

$$\mathbf{a}'\boldsymbol{\mu} = [1, 0, \ldots, 0]\begin{bmatrix}\mu_1\\ \mu_2\\ \vdots\\ \mu_p\end{bmatrix} = \mu_1$$

we have

$$\mathbf{a}'\boldsymbol{\Sigma}\mathbf{a} = [1, 0, \ldots, 0]\begin{bmatrix}\sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p}\\ \sigma_{12} & \sigma_{22} & \cdots & \sigma_{2p}\\ \vdots & \vdots & \ddots & \vdots\\ \sigma_{1p} & \sigma_{2p} & \cdots & \sigma_{pp}\end{bmatrix}\begin{bmatrix}1\\ 0\\ \vdots\\ 0\end{bmatrix} = \sigma_{11}$$

and it follows from Result 4.2 that $X_1$ is distributed as $N(\mu_1, \sigma_{11})$. More generally, the marginal distribution of any component $X_i$ of $\mathbf{X}$ is $N(\mu_i, \sigma_{ii})$. ■

The next result considers several linear combinations of a multivariate normal vector $\mathbf{X}$.

**Result 4.3.** If $\mathbf{X}$ is distributed as $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, the $q$ linear combinations

$$\underset{(q\times p)(p\times 1)}{\mathbf{A}}\ \underset{}{\mathbf{X}} = \begin{bmatrix}a_{11}X_1 + \cdots + a_{1p}X_p\\ a_{21}X_1 + \cdots + a_{2p}X_p\\ \vdots\\ a_{q1}X_1 + \cdots + a_{qp}X_p\end{bmatrix}$$

are distributed as $N_q(\mathbf{A}\boldsymbol{\mu}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}')$. Also, $\underset{(p\times 1)}{\mathbf{X}} + \underset{(p\times 1)}{\mathbf{d}}$, where $\mathbf{d}$ is a vector of constants, is distributed as $N_p(\boldsymbol{\mu} + \mathbf{d}, \boldsymbol{\Sigma})$.

**Proof.** The expected value $E(\mathbf{AX})$ and the covariance matrix of $\mathbf{AX}$ follow from (2–45). Any linear combination $\mathbf{b}'(\mathbf{AX})$ is a linear combination of $\mathbf{X}$, of the form $\mathbf{a}'\mathbf{X}$ with $\mathbf{a} = \mathbf{A}'\mathbf{b}$. Thus, the conclusion concerning $\mathbf{AX}$ follows directly from Result 4.2.

The second part of the result can be obtained by considering $\mathbf{a}'(\mathbf{X} + \mathbf{d}) = \mathbf{a}'\mathbf{X} + (\mathbf{a}'\mathbf{d})$, where $\mathbf{a}'\mathbf{X}$ is distributed as $N(\mathbf{a}'\boldsymbol{\mu}, \mathbf{a}'\boldsymbol{\Sigma}\mathbf{a})$. It is known from the univariate case that adding a constant $\mathbf{a}'\mathbf{d}$ to the random variable $\mathbf{a}'\mathbf{X}$ leaves the variance unchanged and translates the mean to $\mathbf{a}'\boldsymbol{\mu} + \mathbf{a}'\mathbf{d} = \mathbf{a}'(\boldsymbol{\mu} + \mathbf{d})$. Since $\mathbf{a}$ was arbitrary, $\mathbf{X} + \mathbf{d}$ is distributed as $N_p(\boldsymbol{\mu} + \mathbf{d}, \boldsymbol{\Sigma})$. ■

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**Example 4.4 (The distribution of two linear combinations of the components of a normal random vector)** For $\mathbf{X}$ distributed as $N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, find the distribution of

$$\begin{bmatrix}X_1 - X_2\\ X_2 - X_3\end{bmatrix} = \begin{bmatrix}1 & -1 & 0\\ 0 & 1 & -1\end{bmatrix}\begin{bmatrix}X_1\\ X_2\\ X_3\end{bmatrix} = \mathbf{AX}$$

By Result 4.3, the distribution of $\mathbf{AX}$ is multivariate normal with mean

$$\mathbf{A}\boldsymbol{\mu} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = \begin{bmatrix} \mu_1 - \mu_2 \\ \mu_2 - \mu_3 \end{bmatrix}$$

and covariance matrix

$$\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}' = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11} - \sigma_{12} & \sigma_{12} - \sigma_{22} & \sigma_{13} - \sigma_{23} \\ \sigma_{12} - \sigma_{13} & \sigma_{22} - \sigma_{23} & \sigma_{23} - \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11} - 2\sigma_{12} + \sigma_{22} & \sigma_{12} + \sigma_{23} - \sigma_{22} - \sigma_{13} \\ \sigma_{12} + \sigma_{23} - \sigma_{22} - \sigma_{13} & \sigma_{22} - 2\sigma_{23} + \sigma_{33} \end{bmatrix}$$

Alternatively, the mean vector $\mathbf{A}\boldsymbol{\mu}$ and covariance matrix $\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'$ may be verified by direct calculation of the means and covariances of the two random variables $Y_1 = X_1 - X_2$ and $Y_2 = X_2 - X_3$. ■

We have mentioned that all subsets of a multivariate normal random vector $\mathbf{X}$ are themselves normally distributed. We state this property formally as Result 4.4.

**Result 4.4.** All subsets of $\mathbf{X}$ are normally distributed. If we respectively partition $\mathbf{X}$, its mean vector $\boldsymbol{\mu}$, and its covariance matrix $\boldsymbol{\Sigma}$ as

$$\underset{(p\times 1)}{\mathbf{X}} = \begin{bmatrix} \underset{(q\times 1)}{\mathbf{X}_1} \\ \hdashline \underset{((p-q)\times 1)}{\mathbf{X}_2} \end{bmatrix} \qquad \underset{(p\times 1)}{\boldsymbol{\mu}} = \begin{bmatrix} \underset{(q\times 1)}{\boldsymbol{\mu}_1} \\ \hdashline \underset{((p-q)\times 1)}{\boldsymbol{\mu}_2} \end{bmatrix}$$

and

$$\underset{(p\times p)}{\boldsymbol{\Sigma}} = \left[\begin{array}{c:c} \underset{(q\times q)}{\boldsymbol{\Sigma}_{11}} & \underset{(q\times(p-q))}{\boldsymbol{\Sigma}_{12}} \\ \hdashline \underset{((p-q)\times q)}{\boldsymbol{\Sigma}_{21}} & \underset{((p-q)\times(p-q))}{\boldsymbol{\Sigma}_{22}} \end{array}\right]$$

then $\mathbf{X}_1$ is distributed as $N_q(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11})$.

**Proof.** Set $\underset{(q\times p)}{\mathbf{A}} = \left[\underset{(q\times q)}{\mathbf{I}} \;\vdots\; \underset{(q\times(p-q))}{\mathbf{0}}\right]$ in Result 4.3, and the conclusion follows. To apply Result 4.4 to an *arbitrary* subset of the components of $\mathbf{X}$, we simply relabel the subset of interest as $\mathbf{X}_1$ and select the corresponding component means and covariances as $\boldsymbol{\mu}_1$ and $\boldsymbol{\Sigma}_{11}$, respectively. ■

**Example 4.5 (The distribution of a subset of a normal random vector)**

If $\mathbf{X}$ is distributed as $N_5(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, find the distribution of $\begin{bmatrix} X_2 \\ X_4 \end{bmatrix}$. We set

$$\mathbf{X}_1 = \begin{bmatrix} X_2 \\ X_4 \end{bmatrix}, \quad \boldsymbol{\mu}_1 = \begin{bmatrix} \mu_2 \\ \mu_4 \end{bmatrix}, \quad \boldsymbol{\Sigma}_{11} = \begin{bmatrix} \sigma_{22} & \sigma_{24} \\ \sigma_{24} & \sigma_{44} \end{bmatrix}$$

and note that with this assignment, $\mathbf{X}$, $\boldsymbol{\mu}$, and $\boldsymbol{\Sigma}$ can respectively be rearranged and partitioned as

$$\mathbf{X} = \begin{bmatrix} X_2 \\ X_4 \\ \hdashline X_1 \\ X_3 \\ X_5 \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} \mu_2 \\ \mu_4 \\ \hdashline \mu_1 \\ \mu_3 \\ \mu_5 \end{bmatrix}, \quad \boldsymbol{\Sigma} = \left[\begin{array}{cc:ccc} \sigma_{22} & \sigma_{24} & \sigma_{12} & \sigma_{23} & \sigma_{25} \\ \sigma_{24} & \sigma_{44} & \sigma_{14} & \sigma_{34} & \sigma_{45} \\ \hdashline \sigma_{12} & \sigma_{14} & \sigma_{11} & \sigma_{13} & \sigma_{15} \\ \sigma_{23} & \sigma_{34} & \sigma_{13} & \sigma_{33} & \sigma_{35} \\ \sigma_{25} & \sigma_{45} & \sigma_{15} & \sigma_{35} & \sigma_{55} \end{array}\right]$$

or

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ (2\times 1) \\ \hdashline \mathbf{X}_2 \\ (3\times 1) \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ (2\times 1) \\ \hdashline \boldsymbol{\mu}_2 \\ (3\times 1) \end{bmatrix}, \quad \boldsymbol{\Sigma} = \left[\begin{array}{c:c} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ (2\times 2) & (2\times 3) \\ \hdashline \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \\ (3\times 2) & (3\times 3) \end{array}\right]$$

Thus, from Result 4.4, for

$$\mathbf{X}_1 = \begin{bmatrix} X_2 \\ X_4 \end{bmatrix}$$

we have the distribution

$$N_2(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11}) = N_2\left(\begin{bmatrix} \mu_2 \\ \mu_4 \end{bmatrix}, \begin{bmatrix} \sigma_{22} & \sigma_{24} \\ \sigma_{24} & \sigma_{44} \end{bmatrix}\right)$$

It is clear from this example that the normal distribution for any subset can be expressed by simply selecting the appropriate means and covariances from the original $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. The formal process of relabeling and partitioning is unnecessary. ■

We are now in a position to state that zero correlation between normal random variables or sets of normal random variables is equivalent to statistical independence.

**Result 4.5.**

**(a)** If $\underset{(q_1\times 1)}{\mathbf{X}_1}$ and $\underset{(q_2\times 1)}{\mathbf{X}_2}$ are independent, then $\text{Cov}(\mathbf{X}_1, \mathbf{X}_2) = \mathbf{0}$, a $q_1 \times q_2$ matrix of zeros.

**(b)** If $\begin{bmatrix} \mathbf{X}_1 \\ \hdashline \mathbf{X}_2 \end{bmatrix}$ is $N_{q_1+q_2}\left(\begin{bmatrix} \boldsymbol{\mu}_1 \\ \hdashline \boldsymbol{\mu}_2 \end{bmatrix}, \left[\begin{array}{c:c} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \hdashline \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{array}\right]\right)$, then $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent if and only if $\boldsymbol{\Sigma}_{12} = \mathbf{0}$.

(c) If $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent and are distributed as $N_{q_1}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11})$ and $N_{q_2}(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_{22})$, respectively, then $\left[\begin{array}{c}\mathbf{X}_1 \\ \hdashline \mathbf{X}_2\end{array}\right]$ has the multivariate normal distribution

$$N_{q_1+q_2}\left(\left[\begin{array}{c}\boldsymbol{\mu}_1 \\ \hdashline \boldsymbol{\mu}_2\end{array}\right], \left[\begin{array}{c:c}\boldsymbol{\Sigma}_{11} & \mathbf{0} \\ \hdashline \mathbf{0}' & \boldsymbol{\Sigma}_{22}\end{array}\right]\right)$$

**Proof.** (See Exercise 4.14 for partial proofs based upon factoring the density function when $\boldsymbol{\Sigma}_{12} = \mathbf{0}$.) ■

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**Example 4.6 (The equivalence of zero covariance and independence for normal variables)** Let $\underset{(3\times 1)}{\mathbf{X}}$ be $N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with

$$\boldsymbol{\Sigma} = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Are $X_1$ and $X_2$ independent? What about $(X_1, X_2)$ and $X_3$?

Since $X_1$ and $X_2$ have covariance $\sigma_{12} = 1$, they are not independent. However, partitioning $\mathbf{X}$ and $\boldsymbol{\Sigma}$ as

$$\mathbf{X} = \left[\begin{array}{c}X_1 \\ X_2 \\ \hdashline X_3\end{array}\right], \qquad \boldsymbol{\Sigma} = \left[\begin{array}{cc:c}4 & 1 & 0 \\ 1 & 3 & 0 \\ \hdashline 0 & 0 & 2\end{array}\right] = \left[\begin{array}{c:c}\underset{(2\times 2)}{\boldsymbol{\Sigma}_{11}} & \underset{(2\times 1)}{\boldsymbol{\Sigma}_{12}} \\ \hdashline \underset{(1\times 2)}{\boldsymbol{\Sigma}_{21}} & \underset{(1\times 1)}{\boldsymbol{\Sigma}_{22}}\end{array}\right]$$

we see that $\mathbf{X}_1 = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ and $X_3$ have covariance matrix $\boldsymbol{\Sigma}_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Therefore, $(X_1, X_2)$ and $X_3$ are independent by Result 4.5. This implies $X_3$ is independent of $X_1$ and also of $X_2$. ■

We pointed out in our discussion of the bivariate normal distribution that $\rho_{12} = 0$ (zero correlation) implied independence because the joint density function [see (4-6)] could then be written as the product of the marginal (normal) densities of $X_1$ and $X_2$. This fact, which we encouraged you to verify directly, is simply a special case of Result 4.5 with $q_1 = q_2 = 1$.

**Result 4.6.** Let $\mathbf{X} = \left[\begin{array}{c}\mathbf{X}_1 \\ \hdashline \mathbf{X}_2\end{array}\right]$ be distributed as $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\mu} = \left[\begin{array}{c}\boldsymbol{\mu}_1 \\ \hdashline \boldsymbol{\mu}_2\end{array}\right]$, $\boldsymbol{\Sigma} = \left[\begin{array}{c:c}\boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \hdashline \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22}\end{array}\right]$, and $|\boldsymbol{\Sigma}_{22}| > 0$. Then the conditional distribution of $\mathbf{X}_1$, given that $\mathbf{X}_2 = \mathbf{x}_2$, is normal and has

$$\text{Mean} = \boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2)$$

and

$$\text{Covariance} = \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}$$

Note that the covariance does not depend on the value $\mathbf{x}_2$ of the conditioning variable.

**Proof.** We shall give an indirect proof. (See Exercise 4.13, which uses the densities directly.) Take

$$\underset{(p\times p)}{\mathbf{A}} = \left[\begin{array}{c|c} \underset{(q\times q)}{\mathbf{I}} & \underset{q\times(p-q)}{-\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}} \\ \hline \underset{(p-q)\times q}{\mathbf{0}} & \underset{(p-q)\times(p-q)}{\mathbf{I}} \end{array}\right]$$

so

$$\mathbf{A}(\mathbf{X} - \boldsymbol{\mu}) = \mathbf{A}\left[\begin{array}{c} \mathbf{X}_1 - \boldsymbol{\mu}_1 \\ \hline \mathbf{X}_2 - \boldsymbol{\mu}_2 \end{array}\right] = \left[\begin{array}{c} \mathbf{X}_1 - \boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{X}_2 - \boldsymbol{\mu}_2) \\ \hline \mathbf{X}_2 - \boldsymbol{\mu}_2 \end{array}\right]$$

is jointly normal with covariance matrix $\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'$ given by

$$\left[\begin{array}{c|c} \mathbf{I} & -\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1} \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right]\left[\begin{array}{c|c} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \hline \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{array}\right]\left[\begin{array}{c|c} \mathbf{I} & \mathbf{0}' \\ \hline (-\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1})' & \mathbf{I} \end{array}\right] = \left[\begin{array}{c|c} \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21} & \mathbf{0}' \\ \hline \mathbf{0} & \boldsymbol{\Sigma}_{22} \end{array}\right].$$

Since $\mathbf{X}_1 - \boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{X}_2 - \boldsymbol{\mu}_2)$ and $\mathbf{X}_2 - \boldsymbol{\mu}_2$ have zero covariance, they are independent. Moreover, the quantity $\mathbf{X}_1 - \boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{X}_2 - \boldsymbol{\mu}_2)$ has distribution $N_q(\mathbf{0}, \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21})$. Given that $\mathbf{X}_2 = \mathbf{x}_2$, $\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2)$ is a constant. Because $\mathbf{X}_1 - \boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{X}_2 - \boldsymbol{\mu}_2)$ and $\mathbf{X}_2 - \boldsymbol{\mu}_2$ are independent, the conditional distribution of $\mathbf{X}_1 - \boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2)$ is the same as the unconditional distribution of $\mathbf{X}_1 - \boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{X}_2 - \boldsymbol{\mu}_2)$. Since $\mathbf{X}_1 - \boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{X}_2 - \boldsymbol{\mu}_2)$ is $N_q(\mathbf{0}, \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21})$, so is the random vector $\mathbf{X}_1 - \boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2)$ when $\mathbf{X}_2$ has the particular value $\mathbf{x}_2$. Equivalently, given that $\mathbf{X}_2 = \mathbf{x}_2$, $\mathbf{X}_1$ is distributed as $N_q(\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2), \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21})$. ■

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**Example 4.7 (The conditional density of a bivariate normal distribution)** The conditional density of $X_1$, given that $X_2 = x_2$ for any bivariate distribution, is defined by

$$f(x_1|x_2) = \{\text{conditional density of } X_1 \text{ given that } X_2 = x_2\} = \frac{f(x_1, x_2)}{f(x_2)}$$

where $f(x_2)$ is the marginal distribution of $X_2$. If $f(x_1, x_2)$ is the bivariate normal density, show that $f(x_1|x_2)$ is

$$N\left(\mu_1 + \frac{\sigma_{12}}{\sigma_{22}}(x_2 - \mu_2), \sigma_{11} - \frac{\sigma_{12}^2}{\sigma_{22}}\right)$$

Here $\sigma_{11} - \sigma_{12}^2/\sigma_{22} = \sigma_{11}(1 - \rho_{12}^2)$. The two terms involving $x_1 - \mu_1$ in the exponent of the bivariate normal density [see Equation (4-6)] become, apart from the multiplicative constant $-1/2(1 - \rho_{12}^2)$,

$$\frac{(x_1 - \mu_1)^2}{\sigma_{11}} - 2\rho_{12}\frac{(x_1 - \mu_1)(x_2 - \mu_2)}{\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}}$$

$$= \frac{1}{\sigma_{11}}\left[x_1 - \mu_1 - \rho_{12}\frac{\sqrt{\sigma_{11}}}{\sqrt{\sigma_{22}}}(x_2 - \mu_2)\right]^2 - \frac{\rho_{12}^2}{\sigma_{22}}(x_2 - \mu_2)^2$$

Because $\rho_{12} = \sigma_{12}/\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}$, or $\rho_{12}\sqrt{\sigma_{11}}/\sqrt{\sigma_{22}} = \sigma_{12}/\sigma_{22}$, the complete exponent is

$$\frac{-1}{2(1 - \rho_{12}^2)}\left(\frac{(x_1 - \mu_1)^2}{\sigma_{11}} - 2\rho_{12}\frac{(x_1 - \mu_1)(x_2 - \mu_2)}{\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}} + \frac{(x_2 - \mu_2)^2}{\sigma_{22}}\right)$$

$$= \frac{-1}{2\sigma_{11}(1 - \rho_{12}^2)}\left(x_1 - \mu_1 - \rho_{12}\frac{\sqrt{\sigma_{11}}}{\sqrt{\sigma_{22}}}(x_2 - \mu_2)\right)^2$$

$$- \frac{1}{2(1 - \rho_{12}^2)}\left(\frac{1}{\sigma_{22}} - \frac{\rho_{12}^2}{\sigma_{22}}\right)(x_2 - \mu_2)^2$$

$$= \frac{-1}{2\sigma_{11}(1 - \rho_{12}^2)}\left(x_1 - \mu_1 - \frac{\sigma_{12}}{\sigma_{22}}(x_2 - \mu_2)\right)^2 - \frac{1}{2}\frac{(x_2 - \mu_2)^2}{\sigma_{22}}$$

The constant term $2\pi\sqrt{\sigma_{11}\sigma_{22}(1 - \rho_{12}^2)}$ also factors as

$$\sqrt{2\pi}\sqrt{\sigma_{22}} \times \sqrt{2\pi}\sqrt{\sigma_{11}(1 - \rho_{12}^2)}$$

Dividing the joint density of $X_1$ and $X_2$ by the marginal density

$$f(x_2) = \frac{1}{\sqrt{2\pi}\sqrt{\sigma_{22}}}e^{-(x_2 - \mu_2)^2/2\sigma_{22}}$$

and canceling terms yields the conditional density

$$f(x_1|x_2) = \frac{f(x_1, x_2)}{f(x_2)}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{\sigma_{11}(1 - \rho_{12}^2)}}e^{-[x_1 - \mu_1 - (\sigma_{12}/\sigma_{22})(x_2 - \mu_2)]^2/2\sigma_{11}(1 - \rho_{12}^2)},$$

$$-\infty < x_1 < \infty$$

Thus, with our customary notation, the conditional distribution of $X_1$ given that $X_2 = x_2$ is $N(\mu_1 + (\sigma_{12}/\sigma_{22})(x_2 - \mu_2),\ \sigma_{11}(1 - \rho_{12}^2))$. Now, $\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21} = \sigma_{11} - \sigma_{12}^2/\sigma_{22} = \sigma_{11}(1 - \rho_{12}^2)$ and $\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1} = \sigma_{12}/\sigma_{22}$, agreeing with Result 4.6, which we obtained by an indirect method. ■

For the multivariate normal situation, it is worth emphasizing the following:

1. All conditional distributions are (multivariate) normal.
2. The conditional mean is of the form

$$\begin{matrix} \mu_1 + \beta_{1,q+1}(x_{q+1} - \mu_{q+1}) + \cdots + \beta_{1,p}(x_p - \mu_p) \\ \vdots \\ \mu_q + \beta_{q,q+1}(x_{q+1} - \mu_{q+1}) + \cdots + \beta_{q,p}(x_p - \mu_p) \end{matrix} \tag{4-9}$$

where the $\beta$'s are defined by

$$\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1} = \begin{bmatrix} \beta_{1,q+1} & \beta_{1,q+2} & \cdots & \beta_{1,p} \\ \beta_{2,q+1} & \beta_{2,q+2} & \cdots & \beta_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{q,q+1} & \beta_{q,q+2} & \cdots & \beta_{q,p} \end{bmatrix}$$

3. The conditional covariance, $\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}$, does not depend upon the value(s) of the conditioning variable(s).

We conclude this section by presenting two final properties of multivariate normal random vectors. One has to do with the probability content of the ellipsoids of constant density. The other discusses the distribution of another form of linear combinations.

The chi-square distribution determines the variability of the sample variance $s^2 = s_{11}$ for samples from a univariate normal population. It also plays a basic role in the multivariate case.

**Result 4.7.** Let $\mathbf{X}$ be distributed as $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $|\boldsymbol{\Sigma}| > 0$. Then

**(a)** $(\mathbf{X} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X} - \boldsymbol{\mu})$ is distributed as $\chi_p^2$, where $\chi_p^2$ denotes the chi-square distribution with $p$ degrees of freedom.

**(b)** The $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ distribution assigns probability $1 - \alpha$ to the solid ellipsoid $\{\mathbf{x}: (\mathbf{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}) \leq \chi_p^2(\alpha)\}$, where $\chi_p^2(\alpha)$ denotes the upper $(100\alpha)$th percentile of the $\chi_p^2$ distribution.

**Proof.** We know that $\chi_p^2$ is defined as the distribution of the sum $Z_1^2 + Z_2^2 + \cdots + Z_p^2$, where $Z_1, Z_2, \ldots, Z_p$ are independent $N(0, 1)$ random variables. Next, by the spectral decomposition [see Equations (2-16) and (2-21) with $\mathbf{A} = \boldsymbol{\Sigma}$, and see Result 4.1], $\boldsymbol{\Sigma}^{-1} = \sum_{i=1}^{p} \frac{1}{\lambda_i}\mathbf{e}_i\mathbf{e}_i'$, where $\boldsymbol{\Sigma}\mathbf{e}_i = \lambda_i\mathbf{e}_i$, so $\boldsymbol{\Sigma}^{-1}\mathbf{e}_i = (1/\lambda_i)\mathbf{e}_i$. Consequently, $(\mathbf{X} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X} - \boldsymbol{\mu}) = \sum_{i=1}^{p} (1/\lambda_i)(\mathbf{X} - \boldsymbol{\mu})'\mathbf{e}_i\mathbf{e}_i'(\mathbf{X} - \boldsymbol{\mu}) = \sum_{i=1}^{p} (1/\lambda_i)(\mathbf{e}_i'(\mathbf{X} - \boldsymbol{\mu}))^2 = \sum_{i=1}^{p} [(1/\sqrt{\lambda_i})\,\mathbf{e}_i'(\mathbf{X} - \boldsymbol{\mu})]^2 = \sum_{i=1}^{p} Z_i^2$, for instance. Now, we can write $\mathbf{Z} = \mathbf{A}(\mathbf{X} - \boldsymbol{\mu})$,

where

$$\underset{(p\times 1)}{\mathbf{Z}} = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_p \end{bmatrix}, \qquad \underset{(p\times p)}{\mathbf{A}} = \begin{bmatrix} \frac{1}{\sqrt{\lambda_1}}\mathbf{e}_1' \\ \frac{1}{\sqrt{\lambda_2}}\mathbf{e}_2' \\ \vdots \\ \frac{1}{\sqrt{\lambda_p}}\mathbf{e}_p' \end{bmatrix}$$

and $\mathbf{X} - \boldsymbol{\mu}$ is distributed as $N_p(\mathbf{0}, \boldsymbol{\Sigma})$. Therefore, by Result 4.3, $\mathbf{Z} = \mathbf{A}(\mathbf{X} - \boldsymbol{\mu})$ is distributed as $N_p(\mathbf{0}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}')$, where

$$\underset{(p\times p)(p\times p)(p\times p)}{\mathbf{A}\ \boldsymbol{\Sigma}\ \mathbf{A}'} = \begin{bmatrix} \frac{1}{\sqrt{\lambda_1}}\mathbf{e}_1' \\ \frac{1}{\sqrt{\lambda_2}}\mathbf{e}_2' \\ \vdots \\ \frac{1}{\sqrt{\lambda_p}}\mathbf{e}_p' \end{bmatrix} \left[\sum_{i=1}^{p} \lambda_i \mathbf{e}_i \mathbf{e}_i'\right] \left[\frac{1}{\sqrt{\lambda_1}}\mathbf{e}_1 \,\middle|\, \frac{1}{\sqrt{\lambda_2}}\mathbf{e}_2 \,\middle|\, \cdots \,\middle|\, \frac{1}{\sqrt{\lambda_p}}\mathbf{e}_p\right]$$

$$= \begin{bmatrix} \sqrt{\lambda_1}\,\mathbf{e}_1' \\ \sqrt{\lambda_2}\,\mathbf{e}_2' \\ \vdots \\ \sqrt{\lambda_p}\,\mathbf{e}_p' \end{bmatrix} \left[\frac{1}{\sqrt{\lambda_1}}\mathbf{e}_1 \,\middle|\, \frac{1}{\sqrt{\lambda_2}}\mathbf{e}_2 \,\middle|\, \cdots \,\middle|\, \frac{1}{\sqrt{\lambda_p}}\mathbf{e}_p\right] = \mathbf{I}$$

By Result 4.5, $Z_1, Z_2, \ldots, Z_p$ are *independent* standard normal variables, and we conclude that $(\mathbf{X} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X} - \boldsymbol{\mu})$ has a $\chi_p^2$-distribution.

For Part b, we note that $P[(\mathbf{X} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X} - \boldsymbol{\mu}) \le c^2]$ is the probability assigned to the ellipsoid $(\mathbf{X} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X} - \boldsymbol{\mu}) \le c^2$ by the density $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. But from Part a, $P[(\mathbf{X} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X} - \boldsymbol{\mu}) \le \chi_p^2(\alpha)] = 1 - \alpha$, and Part b holds. ■

**Remark: (Interpretation of statistical distance)** Result 4.7 provides an interpretation of a squared statistical distance. When $\mathbf{X}$ is distributed as $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$,

$$(\mathbf{X} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X} - \boldsymbol{\mu})$$

is the squared statistical distance from $\mathbf{X}$ to the population mean vector $\boldsymbol{\mu}$. If one component has a much larger variance than another, it will contribute less to the squared distance. Moreover, two highly correlated random variables will contribute less than two variables that are nearly uncorrelated. Essentially, the use of the inverse of the covariance matrix, (1) standardizes all of the variables and (2) eliminates the effects of correlation. From the proof of Result 4.7,

$$(\mathbf{X} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X} - \boldsymbol{\mu}) = Z_1^2 + Z_2^2 + \cdots + Z_p^2$$

In terms of $\Sigma^{-\frac{1}{2}}$ (see (2-22)), $\mathbf{Z} = \Sigma^{-\frac{1}{2}}(\mathbf{X} - \boldsymbol{\mu})$ has a $N_p(\mathbf{0}, \mathbf{I}_p)$ distribution, and

$$(\mathbf{X} - \boldsymbol{\mu})'\Sigma^{-1}(\mathbf{X} - \boldsymbol{\mu}) = (\mathbf{X} - \boldsymbol{\mu})'\Sigma^{-\frac{1}{2}}\Sigma^{-\frac{1}{2}}(\mathbf{X} - \boldsymbol{\mu})$$
$$= \mathbf{Z}'\mathbf{Z} = Z_1^2 + Z_2^2 + \cdots + Z_p^2$$

The squared statistical distance is calculated as if, first, the random vector $\mathbf{X}$ were transformed to $p$ independent standard normal random variables and then the usual squared distance, the sum of the squares of the variables, were applied.

Next, consider the linear combination of vector random variables

$$c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \cdots + c_n\mathbf{X}_n = \underset{(p\times n)}{[\mathbf{X}_1 \mid \mathbf{X}_2 \mid \cdots \mid \mathbf{X}_n]}\ \underset{(n\times 1)}{\mathbf{c}} \tag{4-10}$$

This linear combination differs from the linear combinations considered earlier in that it defines a $p \times 1$ *vector* random variable that is a linear combination of vectors. Previously, we discussed a *single* random variable that could be written as a linear combination of other univariate random variables.

**Result 4.8.** Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be mutually independent with $\mathbf{X}_j$ distributed as $N_p(\boldsymbol{\mu}_j, \Sigma)$. (Note that each $\mathbf{X}_j$ has the *same* covariance matrix $\Sigma$.) Then

$$\mathbf{V}_1 = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \cdots + c_n\mathbf{X}_n$$

is distributed as $N_p\left(\sum_{j=1}^{n} c_j\boldsymbol{\mu}_j, \left(\sum_{j=1}^{n} c_j^2\right)\Sigma\right)$. Moreover, $\mathbf{V}_1$ and $\mathbf{V}_2 = b_1\mathbf{X}_1 + b_2\mathbf{X}_2 + \cdots + b_n\mathbf{X}_n$ are jointly multivariate normal with covariance matrix

$$\begin{bmatrix} \left(\sum_{j=1}^{n} c_j^2\right)\Sigma & (\mathbf{b}'\mathbf{c})\,\Sigma \\ (\mathbf{b}'\mathbf{c})\,\Sigma & \left(\sum_{j=1}^{n} b_j^2\right)\Sigma \end{bmatrix}$$

Consequently, $\mathbf{V}_1$ and $\mathbf{V}_2$ are independent if $\mathbf{b}'\mathbf{c} = \sum_{j=1}^{n} c_j b_j = 0$.

**Proof.** By Result 4.5(c), the $np$ component vector

$$[X_{11}, \ldots, X_{1p}, X_{21}, \ldots, X_{2p}, \ldots, X_{np}] = [\mathbf{X}_1', \mathbf{X}_2', \ldots, \mathbf{X}_n'] = \underset{(1\times np)}{\mathbf{X}'}$$

is multivariate normal. In particular, $\underset{(np\times 1)}{\mathbf{X}}$ is distributed as $N_{np}(\boldsymbol{\mu}, \Sigma_{\mathbf{x}})$, where

$$\underset{(np\times 1)}{\boldsymbol{\mu}} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \\ \vdots \\ \boldsymbol{\mu}_n \end{bmatrix} \quad \text{and} \quad \underset{(np\times np)}{\Sigma_{\mathbf{x}}} = \begin{bmatrix} \Sigma & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \Sigma & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \Sigma \end{bmatrix}$$

The choice

$$\underset{(2p\times np)}{\mathbf{A}} = \begin{bmatrix} c_1\mathbf{I} & c_2\mathbf{I} & \cdots & c_n\mathbf{I} \\ b_1\mathbf{I} & b_2\mathbf{I} & \cdots & b_n\mathbf{I} \end{bmatrix}$$

where $\mathbf{I}$ is the $p \times p$ identity matrix, gives

$$\mathbf{AX} = \begin{bmatrix} \sum_{j=1}^{n} c_j\mathbf{X}_j \\ \sum_{j=1}^{n} b_j\mathbf{X}_j \end{bmatrix} = \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

and $\mathbf{AX}$ is normal $N_{2p}(\mathbf{A}\boldsymbol{\mu}, \mathbf{A}\boldsymbol{\Sigma}_{\mathbf{x}}\mathbf{A}')$ by Result 4.3. Straightforward block multiplication shows that $\mathbf{A}\boldsymbol{\Sigma}_{\mathbf{x}}\mathbf{A}'$ has the first block diagonal term

$$[c_1\boldsymbol{\Sigma}, c_2\boldsymbol{\Sigma}, \ldots, c_n\boldsymbol{\Sigma}][c_1\mathbf{I}, c_2\mathbf{I}, \ldots, c_n\mathbf{I}]' = \left(\sum_{j=1}^{n} c_j^2\right)\boldsymbol{\Sigma}$$

The off-diagonal term is

$$[c_1\boldsymbol{\Sigma}, c_2\boldsymbol{\Sigma}, \ldots, c_n\boldsymbol{\Sigma}][b_1\mathbf{I}, b_2\mathbf{I}, \ldots, b_n\mathbf{I}]' = \left(\sum_{j=1}^{n} c_jb_j\right)\boldsymbol{\Sigma}$$

This term is the covariance matrix for $\mathbf{V}_1, \mathbf{V}_2$. Consequently, when $\sum_{j=1}^{n} c_jb_j = \mathbf{b}'\mathbf{c} = 0$, so that $\left(\sum_{j=1}^{n} c_jb_j\right)\boldsymbol{\Sigma} = \underset{(p\times p)}{\mathbf{0}}$, $\mathbf{V}_1$ and $\mathbf{V}_2$ are independent by Result 4.5(b). ■

For sums of the type in (4-10), the property of zero correlation is equivalent to requiring the coefficient vectors $\mathbf{b}$ and $\mathbf{c}$ to be perpendicular.

---

**Example 4.8 (Linear combinations of random vectors)** Let $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$, and $\mathbf{X}_4$ be independent and identically distributed $3 \times 1$ random vectors with

$$\boldsymbol{\mu} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

We first consider a linear combination $\mathbf{a}'\mathbf{X}_1$ of the three components of $\mathbf{X}_1$. This is a random variable with mean

$$\mathbf{a}'\boldsymbol{\mu} = 3a_1 - a_2 + a_3$$

and variance

$$\mathbf{a}'\boldsymbol{\Sigma}\,\mathbf{a} = 3a_1^2 + a_2^2 + 2a_3^2 - 2a_1a_2 + 2a_1a_3$$

That is, a linear combination $\mathbf{a}'\mathbf{X}_1$ of the components of a random vector is a single random variable consisting of a sum of terms that are each a constant times a variable. This is very different from a linear combination of random vectors, say,

$$c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + c_3\mathbf{X}_3 + c_4\mathbf{X}_4$$

which is itself a random vector. Here each term in the sum is a constant times a random vector.

Now consider two linear combinations of random vectors

$$\frac{1}{2}\mathbf{X}_1 + \frac{1}{2}\mathbf{X}_2 + \frac{1}{2}\mathbf{X}_3 + \frac{1}{2}\mathbf{X}_4$$

and

$$\mathbf{X}_1 + \mathbf{X}_2 + \mathbf{X}_3 - 3\mathbf{X}_4$$

Find the mean vector and covariance matrix for each linear combination of vectors and also the covariance between them.

By Result 4.8 with $c_1 = c_2 = c_3 = c_4 = 1/2$, the first linear combination has mean vector

$$(c_1 + c_2 + c_3 + c_4)\boldsymbol{\mu} = 2\boldsymbol{\mu} = \begin{bmatrix} 6 \\ -2 \\ 2 \end{bmatrix}$$

and covariance matrix

$$(c_1^2 + c_2^2 + c_3^2 + c_4^2)\boldsymbol{\Sigma} = 1 \times \boldsymbol{\Sigma} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

For the second linear combination of random vectors, we apply Result 4.8 with $b_1 = b_2 = b_3 = 1$ and $b_4 = -3$ to get mean vector

$$(b_1 + b_2 + b_3 + b_4)\boldsymbol{\mu} = 0\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and covariance matrix

$$(b_1^2 + b_2^2 + b_3^2 + b_4^2)\boldsymbol{\Sigma} = 12 \times \boldsymbol{\Sigma} = \begin{bmatrix} 36 & -12 & 12 \\ -12 & 12 & 0 \\ 12 & 0 & 24 \end{bmatrix}$$

Finally, the covariance matrix for the two linear combinations of random vectors is

$$(c_1b_1 + c_2b_2 + c_3b_3 + c_4b_4)\boldsymbol{\Sigma} = 0\boldsymbol{\Sigma} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Every component of the first linear combination of random vectors has zero covariance with every component of the second linear combination of random vectors.

If, in addition, each $\mathbf{X}$ has a trivariate normal distribution, then the two linear combinations have a joint six-variate normal distribution, and the two linear combinations of vectors are independent. ■

# 4.3 Sampling from a Multivariate Normal Distribution and Maximum Likelihood Estimation

We discussed sampling and selecting random samples briefly in Chapter 3. In this section, we shall be concerned with samples from a multivariate normal population—in particular, with the sampling distribution of $\bar{\mathbf{X}}$ and $\mathbf{S}$.

## The Multivariate Normal Likelihood

Let us assume that the $p \times 1$ vectors $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ represent a random sample from a multivariate normal population with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Since $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are mutually independent and each has distribution $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, the joint density function of all the observations is the product of the marginal normal densities:

$$\left\{\begin{matrix}\text{Joint density} \\ \text{of } \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n\end{matrix}\right\} = \prod_{j=1}^{n} \left\{ \frac{1}{(2\pi)^{p/2}|\boldsymbol{\Sigma}|^{1/2}} e^{-(\mathbf{x}_j-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}_j-\boldsymbol{\mu})/2} \right\}$$

$$= \frac{1}{(2\pi)^{np/2}} \frac{1}{|\boldsymbol{\Sigma}|^{n/2}} e^{-\sum_{j=1}^{n} (\mathbf{x}_j-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}_j-\boldsymbol{\mu})/2} \tag{4-11}$$

When the numerical values of the observations become available, they may be substituted for the $\mathbf{x}_j$ in Equation (4-11). The resulting expression, now considered as a function of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ for the fixed set of observations $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, is called the *likelihood*.

Many good statistical procedures employ values for the population parameters that "best" explain the observed data. One meaning of *best* is to select the parameter values that *maximize* the joint density evaluated at the observations. This technique is called *maximum likelihood estimation*, and the maximizing parameter values are called *maximum likelihood estimates*.

At this point, we shall consider maximum likelihood estimation of the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ for a multivariate normal population. To do so, we take the observations $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ as fixed and consider the joint density of Equation (4-11) evaluated at these values. The result is the likelihood function. In order to simplify matters, we rewrite the likelihood function in another form. We shall need some additional properties for the trace of a square matrix. (The trace of a matrix is the sum of its diagonal elements, and the properties of the trace are discussed in Definition 2A.28 and Result 2A.12.)

**Result 4.9.** Let $\mathbf{A}$ be a $k \times k$ symmetric matrix and $\mathbf{x}$ be a $k \times 1$ vector. Then

**(a)** $\mathbf{x}'\mathbf{A}\mathbf{x} = \text{tr}(\mathbf{x}'\mathbf{A}\mathbf{x}) = \text{tr}(\mathbf{A}\mathbf{x}\mathbf{x}')$

**(b)** $\text{tr}(\mathbf{A}) = \sum_{i=1}^{k} \lambda_i$, where the $\lambda_i$ are the eigenvalues of $\mathbf{A}$.

**Proof.** For Part a, we note that $\mathbf{x}'\mathbf{A}\mathbf{x}$ is a scalar, so $\mathbf{x}'\mathbf{A}\mathbf{x} = \text{tr}(\mathbf{x}'\mathbf{A}\mathbf{x})$. We pointed out in Result 2A.12 that $\text{tr}(\mathbf{B}\mathbf{C}) = \text{tr}(\mathbf{C}\mathbf{B})$ for any two matrices $\mathbf{B}$ and $\mathbf{C}$ of dimensions $m \times k$ and $k \times m$, respectively. This follows because $\mathbf{B}\mathbf{C}$ has $\sum_{j=1}^{k} b_{ij}c_{ji}$ as

its $i$th diagonal element, so $\text{tr}(\mathbf{BC}) = \sum_{i=1}^{m}\left(\sum_{j=1}^{k} b_{ij}c_{ji}\right)$. Similarly, the $j$th diagonal element of $\mathbf{CB}$ is $\sum_{i=1}^{m} c_{ji}b_{ij}$, so $\text{tr}(\mathbf{CB}) = \sum_{j=1}^{k}\left(\sum_{i=1}^{m} c_{ji}b_{ij}\right) = \sum_{i=1}^{m}\left(\sum_{j=1}^{k} b_{ij}c_{ji}\right) = \text{tr}(\mathbf{BC})$. Let $\mathbf{x}'$ be the matrix $\mathbf{B}$ with $m = 1$, and let $\mathbf{Ax}$ play the role of the matrix $\mathbf{C}$. Then $\text{tr}(\mathbf{x}'(\mathbf{Ax})) = \text{tr}((\mathbf{Ax})\mathbf{x}')$, and the result follows.

Part b is proved by using the spectral decomposition of (2-20) to write $\mathbf{A} = \mathbf{P}'\mathbf{\Lambda}\mathbf{P}$, where $\mathbf{PP}' = \mathbf{I}$ and $\mathbf{\Lambda}$ is a diagonal matrix with entries $\lambda_1, \lambda_2, \ldots, \lambda_k$. Therefore, $\text{tr}(\mathbf{A}) = \text{tr}(\mathbf{P}'\mathbf{\Lambda}\mathbf{P}) = \text{tr}(\mathbf{\Lambda}\mathbf{PP}') = \text{tr}(\mathbf{\Lambda}) = \lambda_1 + \lambda_2 + \cdots + \lambda_k$. ■

Now the exponent in the joint density in (4–11) can be simplified. By Result 4.9(a),

$$(\mathbf{x}_j - \boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\mathbf{x}_j - \boldsymbol{\mu}) = \text{tr}[(\mathbf{x}_j - \boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\mathbf{x}_j - \boldsymbol{\mu})]$$
$$= \text{tr}[\mathbf{\Sigma}^{-1}(\mathbf{x}_j - \boldsymbol{\mu})(\mathbf{x}_j - \boldsymbol{\mu})'] \tag{4-12}$$

Next,

$$\sum_{j=1}^{n}(\mathbf{x}_j - \boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\mathbf{x}_j - \boldsymbol{\mu}) = \sum_{j=1}^{n}\text{tr}[(\mathbf{x}_j - \boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\mathbf{x}_j - \boldsymbol{\mu})]$$
$$= \sum_{j=1}^{n}\text{tr}[\mathbf{\Sigma}^{-1}(\mathbf{x}_j - \boldsymbol{\mu})(\mathbf{x}_j - \boldsymbol{\mu})']$$
$$= \text{tr}\left[\mathbf{\Sigma}^{-1}\left(\sum_{j=1}^{n}(\mathbf{x}_j - \boldsymbol{\mu})(\mathbf{x}_j - \boldsymbol{\mu})'\right)\right] \tag{4-13}$$

since the trace of a sum of matrices is equal to the sum of the traces of the matrices, according to Result 2A.12(b). We can add and subtract $\bar{\mathbf{x}} = (1/n)\sum_{j=1}^{n}\mathbf{x}_j$ in each term $(\mathbf{x}_j - \boldsymbol{\mu})$ in $\sum_{j=1}^{n}(\mathbf{x}_j - \boldsymbol{\mu})(\mathbf{x}_j - \boldsymbol{\mu})'$ to give

$$\sum_{j=1}^{n}(\mathbf{x}_j - \bar{\mathbf{x}} + \bar{\mathbf{x}} - \boldsymbol{\mu})(\mathbf{x}_j - \bar{\mathbf{x}} + \bar{\mathbf{x}} - \boldsymbol{\mu})'$$
$$= \sum_{j=1}^{n}(\mathbf{x}_j - \bar{\mathbf{x}})(\mathbf{x}_j - \bar{\mathbf{x}})' + \sum_{j=1}^{n}(\bar{\mathbf{x}} - \boldsymbol{\mu})(\bar{\mathbf{x}} - \boldsymbol{\mu})'$$
$$= \sum_{j=1}^{n}(\mathbf{x}_j - \bar{\mathbf{x}})(\mathbf{x}_j - \bar{\mathbf{x}})' + n(\bar{\mathbf{x}} - \boldsymbol{\mu})(\bar{\mathbf{x}} - \boldsymbol{\mu})' \tag{4-14}$$

because the cross-product terms, $\sum_{j=1}^{n}(\mathbf{x}_j - \bar{\mathbf{x}})(\bar{\mathbf{x}} - \boldsymbol{\mu})'$ and $\sum_{j=1}^{n}(\bar{\mathbf{x}} - \boldsymbol{\mu})(\mathbf{x}_j - \bar{\mathbf{x}})'$, are both matrices of zeros. (See Exercise 4.15.) Consequently, using Equations (4-13) and (4-14), we can write the joint density of a random sample from a multivariate normal population as

$$\left\{\begin{matrix}\text{Joint density of}\\ \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n\end{matrix}\right\} = (2\pi)^{-np/2}|\mathbf{\Sigma}|^{-n/2}$$
$$\times \exp\left\{-\text{tr}\left[\mathbf{\Sigma}^{-1}\left(\sum_{j=1}^{n}(\mathbf{x}_j - \bar{\mathbf{x}})(\mathbf{x}_j - \bar{\mathbf{x}})' + n(\bar{\mathbf{x}} - \boldsymbol{\mu})(\bar{\mathbf{x}} - \boldsymbol{\mu})'\right)\right]\Big/2\right\} \tag{4-15}$$

Substituting the observed values $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ into the joint density yields the likelihood function. We shall denote this function by $L(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, to stress the fact that it is a function of the (unknown) population parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. Thus, when the vectors $\mathbf{x}_j$ contain the specific numbers actually observed, we have

$$L(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{np/2}|\boldsymbol{\Sigma}|^{n/2}} e^{-\mathrm{tr}\left[\boldsymbol{\Sigma}^{-1}\left(\sum_{j=1}^{n}(\mathbf{x}_j-\bar{\mathbf{x}})(\mathbf{x}_j-\bar{\mathbf{x}})' + n(\bar{\mathbf{x}}-\boldsymbol{\mu})(\bar{\mathbf{x}}-\boldsymbol{\mu})'\right)\right]/2} \tag{4-16}$$

It will be convenient in later sections of this book to express the exponent in the likelihood function (4-16) in different ways. In particular, we shall make use of the identity

$$\begin{aligned}
&\mathrm{tr}\left[\boldsymbol{\Sigma}^{-1}\left(\sum_{j=1}^{n}(\mathbf{x}_j-\bar{\mathbf{x}})(\mathbf{x}_j-\bar{\mathbf{x}})' + n(\bar{\mathbf{x}}-\boldsymbol{\mu})(\bar{\mathbf{x}}-\boldsymbol{\mu})'\right)\right] \\
&= \mathrm{tr}\left[\boldsymbol{\Sigma}^{-1}\left(\sum_{j=1}^{n}(\mathbf{x}_j-\bar{\mathbf{x}})(\mathbf{x}_j-\bar{\mathbf{x}})'\right)\right] + n\,\mathrm{tr}[\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{x}}-\boldsymbol{\mu})(\bar{\mathbf{x}}-\boldsymbol{\mu})'] \\
&= \mathrm{tr}\left[\boldsymbol{\Sigma}^{-1}\left(\sum_{j=1}^{n}(\mathbf{x}_j-\bar{\mathbf{x}})(\mathbf{x}_j-\bar{\mathbf{x}})'\right)\right] + n(\bar{\mathbf{x}}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{x}}-\boldsymbol{\mu})
\end{aligned} \tag{4-17}$$

## Maximum Likelihood Estimation of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$

The next result will eventually allow us to obtain the maximum likelihood estimators of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$.

**Result 4.10.** Given a $p \times p$ symmetric positive definite matrix $\mathbf{B}$ and a scalar $b > 0$, it follows that

$$\frac{1}{|\boldsymbol{\Sigma}|^b} e^{-\mathrm{tr}(\boldsymbol{\Sigma}^{-1}\mathbf{B})/2} \le \frac{1}{|\mathbf{B}|^b}(2b)^{pb}e^{-bp}$$

for all positive definite $\underset{(p\times p)}{\boldsymbol{\Sigma}}$, with equality holding only for $\boldsymbol{\Sigma} = (1/2b)\mathbf{B}$.

**Proof.** Let $\mathbf{B}^{1/2}$ be the symmetric square root of $\mathbf{B}$ [see Equation (2-22)], so $\mathbf{B}^{1/2}\mathbf{B}^{1/2} = \mathbf{B}$, $\mathbf{B}^{1/2}\mathbf{B}^{-1/2} = \mathbf{I}$, and $\mathbf{B}^{-1/2}\mathbf{B}^{-1/2} = \mathbf{B}^{-1}$. Then $\mathrm{tr}(\boldsymbol{\Sigma}^{-1}\mathbf{B}) = \mathrm{tr}[(\boldsymbol{\Sigma}^{-1}\mathbf{B}^{1/2})\mathbf{B}^{1/2}] = \mathrm{tr}[\mathbf{B}^{1/2}(\boldsymbol{\Sigma}^{-1}\mathbf{B}^{1/2})]$. Let $\eta$ be an eigenvalue of $\mathbf{B}^{1/2}\boldsymbol{\Sigma}^{-1}\mathbf{B}^{1/2}$. This matrix is positive definite because $\mathbf{y}'\mathbf{B}^{1/2}\boldsymbol{\Sigma}^{-1}\mathbf{B}^{1/2}\mathbf{y} = (\mathbf{B}^{1/2}\mathbf{y})'\boldsymbol{\Sigma}^{-1}(\mathbf{B}^{1/2}\mathbf{y}) > 0$ if $\mathbf{B}^{1/2}\mathbf{y} \neq \mathbf{0}$ or, equivalently, $\mathbf{y} \neq \mathbf{0}$. Thus, the eigenvalues $\eta_i$ of $\mathbf{B}^{1/2}\boldsymbol{\Sigma}^{-1}\mathbf{B}^{1/2}$ are positive by Exercise 2.17. Result 4.9(b) then gives

$$\mathrm{tr}(\boldsymbol{\Sigma}^{-1}\mathbf{B}) = \mathrm{tr}(\mathbf{B}^{1/2}\boldsymbol{\Sigma}^{-1}\mathbf{B}^{1/2}) = \sum_{i=1}^{p}\eta_i$$

and $|\mathbf{B}^{1/2}\boldsymbol{\Sigma}^{-1}\mathbf{B}^{1/2}| = \prod_{i=1}^{p}\eta_i$ by Exercise 2.12. From the properties of determinants in Result 2A.11, we can write

$$\begin{aligned}
|\mathbf{B}^{1/2}\boldsymbol{\Sigma}^{-1}\mathbf{B}^{1/2}| &= |\mathbf{B}^{1/2}||\boldsymbol{\Sigma}^{-1}||\mathbf{B}^{1/2}| = |\boldsymbol{\Sigma}^{-1}||\mathbf{B}^{1/2}||\mathbf{B}^{1/2}| \\
&= |\boldsymbol{\Sigma}^{-1}||\mathbf{B}| = \frac{1}{|\boldsymbol{\Sigma}|}|\mathbf{B}|
\end{aligned}$$

or

$$\frac{1}{|\boldsymbol{\Sigma}|} = \frac{|\mathbf{B}^{1/2}\boldsymbol{\Sigma}^{-1}\mathbf{B}^{1/2}|}{|\mathbf{B}|} = \frac{\prod_{i=1}^{p}\eta_i}{|\mathbf{B}|}$$

Combining the results for the trace and the determinant yields

$$\frac{1}{|\boldsymbol{\Sigma}|^b}e^{-\mathrm{tr}[\boldsymbol{\Sigma}^{-1}\mathbf{B}]/2} = \frac{\left(\prod_{i=1}^{p}\eta_i\right)^b}{|\mathbf{B}|^b}e^{-\sum_{i=1}^{p}\eta_i/2} = \frac{1}{|\mathbf{B}|^b}\prod_{i=1}^{p}\eta_i^b e^{-\eta_i/2}$$

But the function $\eta^b e^{-\eta/2}$ has a maximum, with respect to $\eta$, of $(2b)^b e^{-b}$, occurring at $\eta = 2b$. The choice $\eta_i = 2b$, for each $i$, therefore gives

$$\frac{1}{|\boldsymbol{\Sigma}|^b}e^{-\mathrm{tr}(\boldsymbol{\Sigma}^{-1}\mathbf{B})/2} \leq \frac{1}{|\mathbf{B}|^b}(2b)^{pb}e^{-bp}$$

The upper bound is uniquely attained when $\boldsymbol{\Sigma} = (1/2b)\mathbf{B}$, since, for this choice,

$$\mathbf{B}^{1/2}\boldsymbol{\Sigma}^{-1}\mathbf{B}^{1/2} = \mathbf{B}^{1/2}(2b)\mathbf{B}^{-1}\mathbf{B}^{1/2} = (2b)\underset{(p\times p)}{\mathbf{I}}$$

and

$$\mathrm{tr}[\boldsymbol{\Sigma}^{-1}\mathbf{B}] = \mathrm{tr}[\mathbf{B}^{1/2}\boldsymbol{\Sigma}^{-1}\mathbf{B}^{1/2}] = \mathrm{tr}[(2b)\mathbf{I}] = 2bp$$

Moreover,

$$\frac{1}{|\boldsymbol{\Sigma}|} = \frac{|\mathbf{B}^{1/2}\boldsymbol{\Sigma}^{-1}\mathbf{B}^{1/2}|}{|\mathbf{B}|} = \frac{|(2b)\mathbf{I}|}{|\mathbf{B}|} = \frac{(2b)^p}{|\mathbf{B}|}$$

Straightforward substitution for $\mathrm{tr}[\boldsymbol{\Sigma}^{-1}\mathbf{B}]$ and $1/|\boldsymbol{\Sigma}|^b$ yields the bound asserted. ■

The maximum likelihood estimates of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are those values—denoted by $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$—that maximize the function $L(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ in (4-16). The estimates $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$ will depend on the observed values $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ through the summary statistics $\bar{\mathbf{x}}$ and $\mathbf{S}$.

**Result 4.11.** Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be a random sample from a normal population with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$. Then

$$\hat{\boldsymbol{\mu}} = \overline{\mathbf{X}} \quad \text{and} \quad \hat{\boldsymbol{\Sigma}} = \frac{1}{n}\sum_{j=1}^{n}(\mathbf{X}_j - \overline{\mathbf{X}})(\mathbf{X}_j - \overline{\mathbf{X}})' = \frac{(n-1)}{n}\mathbf{S}$$

are the *maximum likelihood estimators* of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, respectively. Their observed values, $\bar{\mathbf{x}}$ and $(1/n)\sum_{j=1}^{n}(\mathbf{x}_j - \bar{\mathbf{x}})(\mathbf{x}_j - \bar{\mathbf{x}})'$, are called the *maximum likelihood estimates* of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$.

**Proof.** The exponent in the likelihood function [see Equation (4-16)], apart from the multiplicative factor $-\frac{1}{2}$, is [see (4-17)]

$$\mathrm{tr}\left[\boldsymbol{\Sigma}^{-1}\left(\sum_{j=1}^{n}(\mathbf{x}_j - \bar{\mathbf{x}})(\mathbf{x}_j - \bar{\mathbf{x}})'\right)\right] + n(\bar{\mathbf{x}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu})$$

By Result 4.1, $\Sigma^{-1}$ is positive definite, so the distance $(\bar{\mathbf{x}} - \boldsymbol{\mu})'\Sigma^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}) > 0$ unless $\boldsymbol{\mu} = \bar{\mathbf{x}}$. Thus, the likelihood is maximized with respect to $\boldsymbol{\mu}$ at $\hat{\boldsymbol{\mu}} = \bar{\mathbf{x}}$. It remains to maximize

$$L(\boldsymbol{\mu}, \Sigma) = \frac{1}{(2\pi)^{np/2}|\Sigma|^{n/2}} e^{-\mathrm{tr}\left[\Sigma^{-1}\left(\sum_{j=1}^{n}(\mathbf{x}_j - \bar{\mathbf{x}})(\mathbf{x}_j - \bar{\mathbf{x}})'\right)\right]/2}$$

over $\Sigma$. By Result 4.10 with $b = n/2$ and $\mathbf{B} = \sum_{j=1}^{n}(\mathbf{x}_j - \bar{\mathbf{x}})(\mathbf{x}_j - \bar{\mathbf{x}})'$, the maximum occurs at $\hat{\Sigma} = (1/n)\sum_{j=1}^{n}(\mathbf{x}_j - \bar{\mathbf{x}})(\mathbf{x}_j - \bar{\mathbf{x}})'$, as stated.

The maximum likelihood estimators are random quantities. They are obtained by replacing the observations $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ in the expressions for $\hat{\boldsymbol{\mu}}$ and $\hat{\Sigma}$ with the corresponding random vectors, $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$. ■

We note that the maximum likelihood estimator $\bar{\mathbf{X}}$ is a random vector and the maximum likelihood estimator $\hat{\Sigma}$ is a random matrix. The maximum likelihood estimates are their particular values for the given data set. In addition, the maximum of the likelihood is

$$L(\hat{\boldsymbol{\mu}}, \hat{\Sigma}) = \frac{1}{(2\pi)^{np/2}} e^{-np/2} \frac{1}{|\hat{\Sigma}|^{n/2}} \tag{4-18}$$

or, since $|\hat{\Sigma}| = [(n-1)/n]^p|\mathbf{S}|$,

$$L(\hat{\boldsymbol{\mu}}, \hat{\Sigma}) = \text{constant} \times (\text{generalized variance})^{-n/2} \tag{4-19}$$

The generalized variance determines the "peakedness" of the likelihood function and, consequently, is a natural measure of variability when the parent population is multivariate normal.

Maximum likelihood estimators possess an *invariance property*. Let $\hat{\boldsymbol{\theta}}$ be the maximum likelihood estimator of $\boldsymbol{\theta}$, and consider estimating the parameter $h(\boldsymbol{\theta})$, which is a function of $\boldsymbol{\theta}$. Then the *maximum likelihood estimate* of

$$\underset{(\text{a function of } \boldsymbol{\theta})}{h(\boldsymbol{\theta})} \quad \text{is given by} \quad \underset{(\text{same function of } \hat{\boldsymbol{\theta}})}{h(\hat{\boldsymbol{\theta}})} \tag{4-20}$$

(See [1] and [15].) For example,

1. The maximum likelihood estimator of $\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu}$ is $\hat{\boldsymbol{\mu}}'\hat{\Sigma}^{-1}\hat{\boldsymbol{\mu}}$, where $\hat{\boldsymbol{\mu}} = \bar{\mathbf{X}}$ and $\hat{\Sigma} = ((n-1)/n)\mathbf{S}$ are the maximum likelihood estimators of $\boldsymbol{\mu}$ and $\Sigma$, respectively.
2. The maximum likelihood estimator of $\sqrt{\sigma_{ii}}$ is $\sqrt{\hat{\sigma}_{ii}}$, where

$$\hat{\sigma}_{ii} = \frac{1}{n}\sum_{j=1}^{n}(X_{ij} - \bar{X}_i)^2$$

is the maximum likelihood estimator of $\sigma_{ii} = \mathrm{Var}(X_i)$.

### Sufficient Statistics

From expression (4-15), the joint density depends on the whole set of observations $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ only through the sample mean $\bar{\mathbf{x}}$ and the sum-of-squares-and-cross-products matrix $\sum_{j=1}^{n} (\mathbf{x}_j - \bar{\mathbf{x}})(\mathbf{x}_j - \bar{\mathbf{x}})' = (n-1)\mathbf{S}$. We express this fact by saying that $\bar{\mathbf{x}}$ and $(n-1)\mathbf{S}$ (or $\mathbf{S}$) are *sufficient statistics*:

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be a random sample from a multivariate normal population with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$. Then

$$\bar{\mathbf{X}} \text{ and } \mathbf{S} \text{ are } \textit{sufficient statistics} \tag{4-21}$$

The importance of sufficient statistics for normal populations is that all of the information about $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ in the data matrix $\mathbf{X}$ is contained in $\bar{\mathbf{x}}$ and $\mathbf{S}$, regardless of the sample size $n$. This generally is not true for nonnormal populations. Since many multivariate techniques begin with sample means and covariances, it is prudent to check on the *adequacy* of the multivariate normal assumption. (See Section 4.6.) If the data cannot be regarded as multivariate normal, techniques that depend solely on $\bar{\mathbf{x}}$ and $\mathbf{S}$ may be ignoring other useful sample information.

## 4.4 The Sampling Distribution of $\bar{X}$ and S

The tentative assumption that $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ constitute a random sample from a normal population with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$ completely determines the sampling distributions of $\bar{\mathbf{X}}$ and $\mathbf{S}$. Here we present the results on the sampling distributions of $\bar{\mathbf{X}}$ and $\mathbf{S}$ by drawing a parallel with the familiar univariate conclusions.

In the univariate case ($p = 1$), we know that $\bar{X}$ is normal with mean $\mu$ = (population mean) and variance

$$\frac{1}{n}\sigma^2 = \frac{\text{population variance}}{\text{sample size}}$$

The result for the multivariate case ($p \geq 2$) is analogous in that $\bar{\mathbf{X}}$ has a normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $(1/n)\boldsymbol{\Sigma}$.

For the sample variance, recall that $(n-1)s^2 = \sum_{j=1}^{n} (X_j - \bar{X})^2$ is distributed as $\sigma^2$ times a chi-square variable having $n-1$ degrees of freedom (d.f.). In turn, this chi-square is the distribution of a sum of squares of independent standard normal random variables. That is, $(n-1)s^2$ is distributed as $\sigma^2(Z_1^2 + \cdots + Z_{n-1}^2) = (\sigma Z_1)^2 + \cdots + (\sigma Z_{n-1})^2$. The individual terms $\sigma Z_i$ are independently distributed as $N(0, \sigma^2)$. It is this latter form that is suitably generalized to the basic sampling distribution for the sample covariance matrix.

The sampling distribution of the sample covariance matrix is called the *Wishart distribution*, after its discoverer; it is defined as the sum of independent products of multivariate normal random vectors. Specifically,

$$W_m(\cdot \mid \boldsymbol{\Sigma}) = \text{Wishart distribution with } m \text{ d.f.} = \text{distribution of } \sum_{j=1}^{m} \mathbf{Z}_j \mathbf{Z}_j' \tag{4-22}$$

where the $\mathbf{Z}_j$ are each independently distributed as $N_p(\mathbf{0}, \boldsymbol{\Sigma})$.

We summarize the sampling distribution results as follows:

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be a random sample of size $n$ from a $p$-variate *normal* distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Then

1. $\bar{\mathbf{X}}$ is distributed as $N_p(\boldsymbol{\mu}, (1/n)\boldsymbol{\Sigma})$.
2. $(n-1)\mathbf{S}$ is distributed as a Wishart random matrix with $n-1$ d.f. (4-23)
3. $\bar{\mathbf{X}}$ and $\mathbf{S}$ are independent.

Because $\boldsymbol{\Sigma}$ is unknown, the distribution of $\bar{\mathbf{X}}$ cannot be used directly to make inferences about $\boldsymbol{\mu}$. However, $\mathbf{S}$ provides independent information about $\boldsymbol{\Sigma}$, and the distribution of $\mathbf{S}$ does not depend on $\boldsymbol{\mu}$. This allows us to construct a statistic for making inferences about $\boldsymbol{\mu}$, as we shall see in Chapter 5.

For the present, we record some further results from multivariable distribution theory. The following properties of the Wishart distribution are derived directly from its definition as a sum of the independent products, $\mathbf{Z}_j \mathbf{Z}_j'$. Proofs can be found in [1].

## Properties of the Wishart Distribution

1. If $\mathbf{A}_1$ is distributed as $W_{m_1}(\mathbf{A}_1 \mid \boldsymbol{\Sigma})$ independently of $\mathbf{A}_2$, which is distributed as $W_{m_2}(\mathbf{A}_2 \mid \boldsymbol{\Sigma})$, then $\mathbf{A}_1 + \mathbf{A}_2$ is distributed as $W_{m_1+m_2}(\mathbf{A}_1 + \mathbf{A}_2 \mid \boldsymbol{\Sigma})$. That is, the degrees of freedom add. (4-24)
2. If $\mathbf{A}$ is distributed as $W_m(\mathbf{A} \mid \boldsymbol{\Sigma})$, then $\mathbf{CAC}'$ is distributed as $W_m(\mathbf{CAC}' \mid \mathbf{C}\boldsymbol{\Sigma}\mathbf{C}')$.

Although we do not have any particular need for the probability density function of the Wishart distribution, it may be of some interest to see its rather complicated form. The density does not exist unless the sample size $n$ is greater than the number of variables $p$. When it does exist, its value at the positive definite matrix $\mathbf{A}$ is

$$w_{n-1}(\mathbf{A} \mid \boldsymbol{\Sigma}) = \frac{|\mathbf{A}|^{(n-p-2)/2} e^{-\operatorname{tr}[\mathbf{A}\boldsymbol{\Sigma}^{-1}]/2}}{2^{p(n-1)/2} \pi^{p(p-1)/4} |\boldsymbol{\Sigma}|^{(n-1)/2} \prod_{i=1}^{p} \Gamma\left(\frac{1}{2}(n-i)\right)}, \quad \mathbf{A} \text{ positive definite} \tag{4-25}$$

where $\Gamma(\cdot)$ is the gamma function. (See [1] and [11].)

## 4.5 Large-Sample Behavior of $\bar{X}$ and S

Suppose the quantity $X$ is determined by a large number of independent causes $V_1, V_2, \ldots, V_n$, where the random variables $V_i$ representing the causes have approximately the same variability. If $X$ is the sum

$$X = V_1 + V_2 + \cdots + V_n$$

then the central limit theorem applies, and we conclude that $X$ has a distribution that is nearly normal. This is true for virtually any parent distribution of the $V_i$'s, provided that $n$ is large enough.

The univariate central limit theorem also tells us that the sampling distribution of the sample mean, $\bar{X}$ for a large sample size is nearly normal, whatever the form of the underlying population distribution. A similar result holds for many other important univariate statistics.

It turns out that certain multivariate statistics, like $\bar{\mathbf{X}}$ and $\mathbf{S}$, have large-sample properties analogous to their univariate counterparts. As the sample size is increased without bound, certain regularities govern the sampling variation in $\bar{\mathbf{X}}$ and $\mathbf{S}$, irrespective of the form of the parent population. Therefore, the conclusions presented in this section do not require multivariate normal populations. The only requirements are that the parent population, whatever its form, have a mean $\boldsymbol{\mu}$ and a finite covariance $\boldsymbol{\Sigma}$.

**Result 4.12 (Law of large numbers).** Let $Y_1, Y_2, \ldots, Y_n$ be independent observations from a population with mean $E(Y_i) = \mu$. Then

$$\bar{Y} = \frac{Y_1 + Y_2 + \cdots + Y_n}{n}$$

converges in probability to $\mu$ as $n$ increases without bound. That is, for any prescribed accuracy $\varepsilon > 0$, $P[-\varepsilon < \bar{Y} - \mu < \varepsilon]$ approaches unity as $n \to \infty$.

**Proof.** See [9]. ■

As a direct consequence of the law of large numbers, which says that each $\bar{X}_i$ converges in probability to $\mu_i$, $i = 1, 2, \ldots, p$,

$$\bar{\mathbf{X}} \text{ converges in probability to } \boldsymbol{\mu} \tag{4-26}$$

Also, each sample covariance $s_{ik}$ converges in probability to $\sigma_{ik}$, $i, k = 1, 2, \ldots, p$, and

$$\mathbf{S} \text{ (or } \hat{\boldsymbol{\Sigma}} = \mathbf{S}_n\text{) converges in probability to } \boldsymbol{\Sigma} \tag{4-27}$$

Statement (4-27) follows from writing

$$\begin{aligned}(n-1)s_{ik} &= \sum_{j=1}^{n} (X_{ji} - \bar{X}_i)(X_{jk} - \bar{X}_k) \\ &= \sum_{j=1}^{n} (X_{ji} - \mu_i + \mu_i - \bar{X}_i)(X_{jk} - \mu_k + \mu_k - \bar{X}_k) \\ &= \sum_{j=1}^{n} (X_{ji} - \mu_i)(X_{jk} - \mu_k) + n(\bar{X}_i - \mu_i)(\bar{X}_k - \mu_k)\end{aligned}$$

Letting $Y_j = (X_{ji} - \mu_i)(X_{jk} - \mu_k)$, with $E(Y_j) = \sigma_{ik}$, we see that the first term in $s_{ik}$ converges to $\sigma_{ik}$ and the second term converges to zero, by applying the law of large numbers.

The practical interpretation of statements (4-26) and (4-27) is that, with high probability, $\bar{\mathbf{X}}$ will be close to $\boldsymbol{\mu}$ and $\mathbf{S}$ will be close to $\boldsymbol{\Sigma}$ whenever the sample size is large. The statement concerning $\bar{\mathbf{X}}$ is made even more precise by a multivariate version of the central limit theorem.

**Result 4.13 (The central limit theorem).** Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be independent observations from any population with mean $\boldsymbol{\mu}$ and finite covariance $\boldsymbol{\Sigma}$. Then

$$\sqrt{n}\,(\bar{\mathbf{X}} - \boldsymbol{\mu}) \text{ has an approximate } N_p(\mathbf{0}, \boldsymbol{\Sigma}) \text{ distribution}$$

for large sample sizes. Here $n$ should also be large relative to $p$.

**Proof.** See [1]. ■

The approximation provided by the central limit theorem applies to discrete, as well as continuous, multivariate populations. Mathematically, the limit is exact, and the approach to normality is often fairly rapid. Moreover, from the results in Section 4.4, we know that $\bar{\mathbf{X}}$ is exactly normally distributed when the underlying population is normal. Thus, we would expect the central limit theorem approximation to be quite good for moderate $n$ when the parent population is nearly normal.

As we have seen, when $n$ is large, $\mathbf{S}$ is close to $\boldsymbol{\Sigma}$ with high probability. Consequently, replacing $\boldsymbol{\Sigma}$ by $\mathbf{S}$ in the approximating normal distribution for $\bar{\mathbf{X}}$ will have a negligible effect on subsequent probability calculations.

Result 4.7 can be used to show that $n(\bar{\mathbf{X}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{X}} - \boldsymbol{\mu})$ has a $\chi^2_p$ distribution when $\bar{\mathbf{X}}$ is distributed as $N_p\left(\boldsymbol{\mu}, \frac{1}{n}\boldsymbol{\Sigma}\right)$ or, equivalently, when $\sqrt{n}\,(\bar{\mathbf{X}} - \boldsymbol{\mu})$ has an $N_p(\mathbf{0}, \boldsymbol{\Sigma})$ distribution. The $\chi^2_p$ distribution is *approximately* the sampling distribution of $n(\bar{\mathbf{X}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{X}} - \boldsymbol{\mu})$ when $\bar{\mathbf{X}}$ is approximately normally distributed. Replacing $\boldsymbol{\Sigma}^{-1}$ by $\mathbf{S}^{-1}$ does not seriously affect this approximation for $n$ large and much greater than $p$.

We summarize the major conclusions of this section as follows:

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be independent observations from a population with mean $\boldsymbol{\mu}$ and finite (nonsingular) covariance $\boldsymbol{\Sigma}$. Then

$$\sqrt{n}\,(\bar{\mathbf{X}} - \boldsymbol{\mu}) \text{ is approximately } N_p(\mathbf{0}, \boldsymbol{\Sigma})$$

and (4-28)

$$n(\bar{\mathbf{X}} - \boldsymbol{\mu})'\mathbf{S}^{-1}(\bar{\mathbf{X}} - \boldsymbol{\mu}) \text{ is approximately } \chi^2_p$$

for $n - p$ large.

In the next three sections, we consider ways of verifying the assumption of normality and methods for transforming nonnormal observations into observations that are approximately normal.

## 4.6 Assessing the Assumption of Normality

As we have pointed out, most of the statistical techniques discussed in subsequent chapters assume that each vector observation $\mathbf{X}_j$ comes from a multivariate normal distribution. On the other hand, in situations where the sample size is large and the techniques depend solely on the behavior of $\bar{\mathbf{X}}$, or distances involving $\bar{\mathbf{X}}$ of the form $n(\bar{\mathbf{X}} - \boldsymbol{\mu})'\mathbf{S}^{-1}(\bar{\mathbf{X}} - \boldsymbol{\mu})$, the assumption of normality for the individual observations is less crucial. But to some degree, the *quality* of inferences made by these methods depends on how closely the true parent population resembles the multivariate normal form. It is imperative, then, that procedures exist for detecting cases where the data exhibit moderate to extreme departures from what is expected under multivariate normality.

We want to answer this question: Do the observations $\mathbf{X}_j$ appear to violate the assumption that they came from a normal population? Based on the properties of normal distributions, we know that all linear combinations of normal variables are normal and the contours of the multivariate normal density are ellipsoids. Therefore, we address these questions:

1. Do the marginal distributions of the elements of $\mathbf{X}$ appear to be normal? What about a few linear combinations of the components $X_i$?
2. Do the scatter plots of pairs of observations on different characteristics give the elliptical appearance expected from normal populations?
3. Are there any "wild" observations that should be checked for accuracy?

It will become clear that our investigations of normality will concentrate on the behavior of the observations in one or two dimensions (for example, marginal distributions and scatter plots). As might be expected, it has proved difficult to construct a "good" overall test of joint normality in more than two dimensions because of the large number of things that can go wrong. To some extent, we must pay a price for concentrating on univariate and bivariate examinations of normality: We can never be sure that we have not missed some feature that is revealed only in higher dimensions. (It is possible, for example, to construct a nonnormal bivariate distribution with normal marginals. [See Exercise 4.8.]) Yet many types of nonnormality are often reflected in the marginal distributions and scatter plots. Moreover, for most practical work, one-dimensional and two-dimensional investigations are ordinarily sufficient. Fortunately, pathological data sets that are normal in lower dimensional representations, but nonnormal in higher dimensions, are not frequently encountered in practice.

### Evaluating the Normality of the Univariate Marginal Distributions

Dot diagrams for smaller $n$ and histograms for $n > 25$ or so help reveal situations where one tail of a univariate distribution is much longer than the other. If the histogram for a variable $X_i$ appears reasonably symmetric, we can check further by counting the number of observations in certain intervals. A univariate normal distribution assigns probability .683 to the interval $(\mu_i - \sqrt{\sigma_{ii}}, \mu_i + \sqrt{\sigma_{ii}})$ and probability .954 to the interval $(\mu_i - 2\sqrt{\sigma_{ii}}, \mu_i + 2\sqrt{\sigma_{ii}})$. Consequently, with a large sample size $n$, we expect the observed proportion $\hat{p}_{i1}$ of the observations lying in the

interval $(\bar{x}_i - \sqrt{s_{ii}}, \bar{x}_i + \sqrt{s_{ii}})$ to be about .683. Similarly, the observed proportion $\hat{p}_{i2}$ of the observations in $(\bar{x}_i - 2\sqrt{s_{ii}}, \bar{x}_i + 2\sqrt{s_{ii}})$ should be about .954. Using the normal approximation to the sampling distribution of $\hat{p}_i$ (see [9]), we observe that either

$$|\hat{p}_{i1} - .683| > 3\sqrt{\frac{(.683)(.317)}{n}} = \frac{1.396}{\sqrt{n}}$$

or

$$|\hat{p}_{i2} - .954| > 3\sqrt{\frac{(.954)(.046)}{n}} = \frac{.628}{\sqrt{n}} \tag{4-29}$$

would indicate departures from an assumed normal distribution for the $i$th characteristic. When the observed proportions are too small, parent distributions with thicker tails than the normal are suggested.

Plots are always useful devices in any data analysis. Special plots called *Q–Q plots* can be used to assess the assumption of normality. These plots can be made for the marginal distributions of the sample observations on each variable. They are, in effect, plots of the sample quantile versus the quantile one would expect to observe if the observations actually were normally distributed. When the points lie very nearly along a straight line, the normality assumption remains tenable. Normality is suspect if the points deviate from a straight line. Moreover, the pattern of the deviations can provide clues about the nature of the nonnormality. Once the reasons for the nonnormality are identified, corrective action is often possible. (See Section 4.8.)

To simplify notation, let $x_1, x_2, \ldots, x_n$ represent $n$ observations on any single characteristic $X_i$. Let $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ represent these observations after they are ordered according to magnitude. For example, $x_{(2)}$ is the second smallest observation and $x_{(n)}$ is the largest observation. The $x_{(j)}$'s are the sample quantiles. When the $x_{(j)}$ are distinct, exactly $j$ observations are less than or equal to $x_{(j)}$. (This is theoretically always true when the observations are of the continuous type, which we usually assume.) The proportion $j/n$ of the sample at or to the left of $x_{(j)}$ is often approximated by $\left(j - \frac{1}{2}\right)/n$ for analytical convenience.[1]

For a standard normal distribution, the quantiles $q_{(j)}$ are defined by the relation

$$P[Z \leq q_{(j)}] = \int_{-\infty}^{q(j)} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz = p_{(j)} = \frac{j - \frac{1}{2}}{n} \tag{4-30}$$

(See Table 1 in the appendix). Here $p_{(j)}$ is the probability of getting a value less than or equal to $q_{(j)}$ in a single drawing from a standard normal population.

The idea is to look at the pairs of quantiles $(q_{(j)}, x_{(j)})$ with the same associated cumulative probability $\left(j - \frac{1}{2}\right)/n$. If the data arise from a normal population, the pairs $(q_{(j)}, x_{(j)})$ will be approximately linearly related, since $\sigma q_{(j)} + \mu$ is nearly the expected sample quantile.[2]

[1] The $\frac{1}{2}$ in the numerator of $\left(j - \frac{1}{2}\right)/n$ is a "continuity" correction. Some authors (see [5] and [10]) have suggested replacing $\left(j - \frac{1}{2}\right)/n$ by $\left(j - \frac{3}{8}\right)/\left(n + \frac{1}{4}\right)$.

[2] A better procedure is to plot $(m_{(j)}, x_{(j)})$, where $m_{(j)} = E(z_{(j)})$ is the expected value of the $j$th-order statistic in a sample of size $n$ from a standard normal distribution. (See [13] for further discussion.)

**Example 4.9 (Constructing a Q–Q plot)** A sample of $n = 10$ observations gives the values in the following table:

| Ordered observations $x_{(j)}$ | Probability levels $(j - \frac{1}{2})/n$ | Standard normal quantiles $q_{(j)}$ |
|---|---|---|
| −1.00 | .05 | −1.645 |
| −.10 | .15 | −1.036 |
| .16 | .25 | −.674 |
| .41 | .35 | −.385 |
| .62 | .45 | −.125 |
| .80 | .55 | .125 |
| 1.26 | .65 | .385 |
| 1.54 | .75 | .674 |
| 1.71 | .85 | 1.036 |
| 2.30 | .95 | 1.645 |

Here, for example, $P[Z \leq .385] = \int_{-\infty}^{.385} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz = .65$. [See (4-30).]

Let us now construct the $Q$–$Q$ plot and comment on its appearance. The $Q$–$Q$ plot for the foregoing data, which is a plot of the ordered data $x_{(j)}$ against the normal quantiles $q_{(j)}$, is shown in Figure 4.5. The pairs of points $(q_{(j)}, x_{(j)})$ lie very nearly along a straight line, and we would not reject the notion that these data are normally distributed—particularly with a sample size as small as $n = 10$.

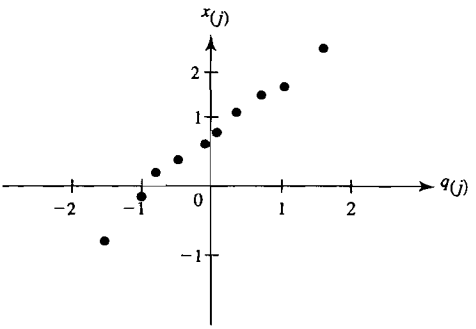


**Figure 4.5** A $Q$–$Q$ plot for the data in Example 4.9. ■

The calculations required for $Q$–$Q$ plots are easily programmed for electronic computers. Many statistical programs available commercially are capable of producing such plots.

The steps leading to a $Q$–$Q$ plot are as follows:

1. Order the original observations to get $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ and their corresponding probability values $(1 - \frac{1}{2})/n, (2 - \frac{1}{2})/n, \ldots, (n - \frac{1}{2})/n$;
2. Calculate the standard normal quantiles $q_{(1)}, q_{(2)}, \ldots, q_{(n)}$; and
3. Plot the pairs of observations $(q_{(1)}, x_{(1)}), (q_{(2)}, x_{(2)}), \ldots, (q_{(n)}, x_{(n)})$, and examine the "straightness" of the outcome.

$Q$–$Q$ plots are not particularly informative unless the sample size is moderate to large—for instance, $n \geq 20$. There can be quite a bit of variability in the straightness of the $Q$–$Q$ plot for small samples, even when the observations are known to come from a normal population.

**Example 4.10 (A Q-Q plot for radiation data)** The quality-control department of a manufacturer of microwave ovens is required by the federal government to monitor the amount of radiation emitted when the doors of the ovens are closed. Observations of the radiation emitted through closed doors of $n = 42$ randomly selected ovens were made. The data are listed in Table 4.1.

**Table 4.1** Radiation Data (Door Closed)

| Oven no. | Radiation | Oven no. | Radiation | Oven no. | Radiation |
|---|---|---|---|---|---|
| 1 | .15 | 16 | .10 | 31 | .10 |
| 2 | .09 | 17 | .02 | 32 | .20 |
| 3 | .18 | 18 | .10 | 33 | .11 |
| 4 | .10 | 19 | .01 | 34 | .30 |
| 5 | .05 | 20 | .40 | 35 | .02 |
| 6 | .12 | 21 | .10 | 36 | .20 |
| 7 | .08 | 22 | .05 | 37 | .20 |
| 8 | .05 | 23 | .03 | 38 | .30 |
| 9 | .08 | 24 | .05 | 39 | .30 |
| 10 | .10 | 25 | .15 | 40 | .40 |
| 11 | .07 | 26 | .10 | 41 | .30 |
| 12 | .02 | 27 | .15 | 42 | .05 |
| 13 | .01 | 28 | .09 | | |
| 14 | .10 | 29 | .08 | | |
| 15 | .10 | 30 | .18 | | |

Source: Data courtesy of J. D. Cryer.

In order to determine the probability of exceeding a prespecified tolerance level, a probability distribution for the radiation emitted was needed. Can we regard the observations here as being normally distributed?

A computer was used to assemble the pairs $(q_{(j)}, x_{(j)})$ and construct the $Q$–$Q$ plot, pictured in Figure 4.6 on page 181. It appears from the plot that the data as a whole are not normally distributed. The points indicated by the circled locations in the figure are outliers—values that are too large relative to the rest of the observations.

For the radiation data, several observations are equal. When this occurs, those observations with like values are associated with the same normal quantile. This quantile is calculated using the average of the quantiles the tied observations would have if they all differed slightly. ■

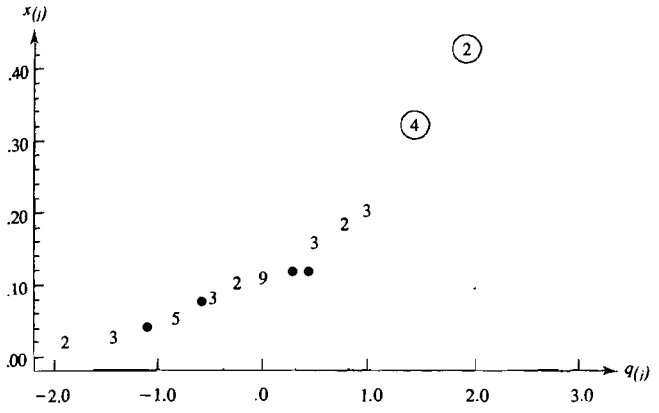


**Figure 4.6** A $Q$–$Q$ plot of the radiation data (door closed) from Example 4.10. (The integers in the plot indicate the number of points occupying the same location.)

The straightness of the $Q$–$Q$ plot can be measured by calculating the correlation coefficient of the points in the plot. The correlation coefficient for the $Q$–$Q$ plot is defined by

$$r_Q = \frac{\sum_{j=1}^{n} (x_{(j)} - \bar{x})(q_{(j)} - \bar{q})}{\sqrt{\sum_{j=1}^{n} (x_{(j)} - \bar{x})^2} \sqrt{\sum_{j=1}^{n} (q_{(j)} - \bar{q})^2}} \tag{4-31}$$

and a powerful test of normality can be based on it. (See [5], [10], and [12].) Formally, we reject the hypothesis of normality at level of significance $\alpha$ if $r_Q$ falls *below* the appropriate value in Table 4.2.

**Table 4.2** Critical Points for the Q–Q Plot Correlation Coefficient Test for Normality

| Sample size $n$ | Significance levels $\alpha$ | | |
|---|---|---|---|
| | .01 | .05 | .10 |
| 5 | .8299 | .8788 | .9032 |
| 10 | .8801 | .9198 | .9351 |
| 15 | .9126 | .9389 | .9503 |
| 20 | .9269 | .9508 | .9604 |
| 25 | .9410 | .9591 | .9665 |
| 30 | .9479 | .9652 | .9715 |
| 35 | .9538 | .9682 | .9740 |
| 40 | .9599 | .9726 | .9771 |
| 45 | .9632 | .9749 | .9792 |
| 50 | .9671 | .9768 | .9809 |
| 55 | .9695 | .9787 | .9822 |
| 60 | .9720 | .9801 | .9836 |
| 75 | .9771 | .9838 | .9866 |
| 100 | .9822 | .9873 | .9895 |
| 150 | .9879 | .9913 | .9928 |
| 200 | .9905 | .9931 | .9942 |
| 300 | .9935 | .9953 | .9960 |

**Example 4.11 (A correlation coefficient test for normality)** Let us calculate the correlation coefficient $r_Q$ from the $Q$–$Q$ plot of Example 4.9 (see Figure 4.5) and test for normality.

Using the information from Example 4.9, we have $\bar{x} = .770$ and

$$\sum_{j=1}^{10} (x_{(j)} - \bar{x})q_{(j)} = 8.584, \quad \sum_{j=1}^{10} (x_{(j)} - \bar{x})^2 = 8.472, \quad \text{and} \quad \sum_{j=1}^{10} q_{(j)}^2 = 8.795$$

Since always, $\bar{q} = 0$,

$$r_Q = \frac{8.584}{\sqrt{8.472}\sqrt{8.795}} = .994$$

A test of normality at the 10% level of significance is provided by referring $r_Q = .994$ to the entry in Table 4.2 corresponding to $n = 10$ and $\alpha = .10$. This entry is .9351. Since $r_Q > .9351$, we do not reject the hypothesis of normality. ■

Instead of $r_Q$, some software packages evaluate the original statistic proposed by Shapiro and Wilk [12]. Its correlation form corresponds to replacing $q_{(j)}$ by a function of the expected value of standard normal-order statistics and their covariances. We prefer $r_Q$ because it corresponds directly to the points in the normal-scores plot. For large sample sizes, the two statistics are nearly the same (see [13]), so either can be used to judge lack of fit.

Linear combinations of more than one characteristic can be investigated. Many statisticians suggest plotting

$$\hat{\mathbf{e}}_1'\mathbf{x}_j \quad \text{where} \quad \mathbf{S}\hat{\mathbf{e}}_1 = \hat{\lambda}_1\hat{\mathbf{e}}_1$$

in which $\hat{\lambda}_1$ is the largest eigenvalue of $\mathbf{S}$. Here $\mathbf{x}_j' = [x_{j1}, x_{j2}, \ldots, x_{jp}]$ is the $j$th observation on the $p$ variables $X_1, X_2, \ldots, X_p$. The linear combination $\hat{\mathbf{e}}_p'\mathbf{x}_j$ corresponding to the smallest eigenvalue is also frequently singled out for inspection. (See Chapter 8 and [6] for further details.)

## Evaluating Bivariate Normality

We would like to check on the assumption of normality for all distributions of $2, 3, \ldots, p$ dimensions. However, as we have pointed out, for practical work it is usually sufficient to investigate the univariate and bivariate distributions. We considered univariate marginal distributions earlier. It is now of interest to examine the bivariate case.

In Chapter 1, we described scatter plots for pairs of characteristics. If the observations were generated from a multivariate normal distribution, each bivariate distribution would be normal, and the contours of constant density would be ellipses. The scatter plot should conform to this structure by exhibiting an overall pattern that is nearly elliptical.

Moreover, by Result 4.7, the set of bivariate outcomes $\mathbf{x}$ such that

$$(\mathbf{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}) \leq \chi_2^2(.5)$$

has probability .5. Thus, we should expect *roughly* the same percentage, 50%, of sample observations to lie in the ellipse given by

$$\{\text{all } \mathbf{x} \text{ such that } (\mathbf{x} - \bar{\mathbf{x}})'\mathbf{S}^{-1}(\mathbf{x} - \bar{\mathbf{x}}) \leq \chi_2^2(.5)\}$$

where we have replaced $\boldsymbol{\mu}$ by its estimate $\bar{\mathbf{x}}$ and $\boldsymbol{\Sigma}^{-1}$ by its estimate $\mathbf{S}^{-1}$. If not, the normality assumption is suspect.

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**Example 4.12 (Checking bivariate normality)** Although not a random sample, data consisting of the pairs of observations ($x_1$ = sales, $x_2$ = profits) for the 10 largest companies in the world are listed in Exercise 1.4. These data give

$$\bar{\mathbf{x}} = \begin{bmatrix} 155.60 \\ 14.70 \end{bmatrix}, \qquad \mathbf{S} = \begin{bmatrix} 7476.45 & 303.62 \\ 303.62 & 26.19 \end{bmatrix}$$

so

$$\mathbf{S}^{-1} = \frac{1}{103{,}623.12}\begin{bmatrix} 26.19 & -303.62 \\ -303.62 & 7476.45 \end{bmatrix}$$

$$= \begin{bmatrix} .000253 & -.002930 \\ -.002930 & .072148 \end{bmatrix}$$

From Table 3 in the appendix, $\chi_2^2(.5) = 1.39$. Thus, any observation $\mathbf{x}' = [x_1, x_2]$ satisfying

$$\begin{bmatrix} x_1 - 155.60 \\ x_2 - 14.70 \end{bmatrix}' \begin{bmatrix} .000253 & -.002930 \\ -.002930 & .072148 \end{bmatrix} \begin{bmatrix} x_1 - 155.60 \\ x_2 - 14.70 \end{bmatrix} \leq 1.39$$

is on or inside the estimated 50% contour. Otherwise the observation is outside this contour. The first pair of observations in Exercise 1.4 is $[x_1, x_2]' = [108.28, 17.05]$. In this case

$$\begin{bmatrix} 108.28 - 155.60 \\ 17.05 - 14.70 \end{bmatrix}' \begin{bmatrix} .000253 & -.002930 \\ -.002930 & .072148 \end{bmatrix} \begin{bmatrix} 108.28 - 155.60 \\ 17.05 - 14.70 \end{bmatrix}$$

$$= 1.61 > 1.39$$

and this point falls outside the 50% contour. The remaining nine points have generalized distances from $\bar{\mathbf{x}}$ of .30, .62, 1.79, 1.30, 4.38, 1.64, 3.53, 1.71, and 1.16, respectively. Since four of these distances are less than 1.39, a proportion, .40, of the data falls within the 50% contour. If the observations were normally distributed, we would expect about half, or 5, of them to be within this contour. This difference in proportions might ordinarily provide evidence for rejecting the notion of bivariate normality; however, our sample size of 10 is too small to reach this conclusion. (See also Example 4.13.) ■

Computing the fraction of the points within a contour and subjectively comparing it with the theoretical probability is a useful, but rather rough, procedure.

A somewhat more formal method for judging the joint normality of a data set is based on the squared generalized distances

$$d_j^2 = (\mathbf{x}_j - \bar{\mathbf{x}})'\mathbf{S}^{-1}(\mathbf{x}_j - \bar{\mathbf{x}}), \qquad j = 1, 2, \ldots, n \tag{4-32}$$

where $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are the sample observations. The procedure we are about to describe is not limited to the bivariate case; it can be used for all $p \geq 2$.

When the parent population is multivariate normal and both $n$ and $n - p$ are greater than 25 or 30, each of the squared distances $d_1^2, d_2^2, \ldots, d_n^2$ should behave like a chi-square random variable. [See Result 4.7 and Equations (4-26) and (4-27).] Although these distances are *not* independent or exactly chi-square distributed, it is helpful to plot them as if they were. The resulting plot is called a *chi-square plot* or *gamma plot*, because the chi-square distribution is a special case of the more general gamma distribution. (See [6].)

To construct the chi-square plot,

1. Order the squared distances in (4-32) from smallest to largest as $d_{(1)}^2 \leq d_{(2)}^2 \leq \cdots \leq d_{(n)}^2$.
2. Graph the pairs $\left(q_{c,p}\left(\left(j - \frac{1}{2}\right)/n\right), d_{(j)}^2\right)$, where $q_{c,p}\left(\left(j - \frac{1}{2}\right)/n\right)$ is the $100\left(j - \frac{1}{2}\right)/n$ quantile of the chi-square distribution with $p$ degrees of freedom.

Quantiles are specified in terms of proportions, whereas percentiles are specified in terms of percentages.

The quantiles $q_{c,p}\left(\left(j - \frac{1}{2}\right)/n\right)$ are related to the upper percentiles of a chi-squared distribution. In particular, $q_{c,p}\left(\left(j - \frac{1}{2}\right)/n\right) = \chi_p^2\left(\left(n - j + \frac{1}{2}\right)/n\right)$.

The plot should resemble a straight line through the origin having slope 1. A systematic curved pattern suggests lack of normality. One or two points far above the line indicate large distances, or outlying observations, that merit further attention.

**Example 4.13 (Constructing a chi-square plot)** Let us construct a chi-square plot of the generalized distances given in Example 4.12. The ordered distances and the corresponding chi-square percentiles for $p = 2$ and $n = 10$ are listed in the following table:

| $j$ | $d_{(j)}^2$ | $q_{c,2}\left(\frac{j - \frac{1}{2}}{10}\right)$ |
|---|---|---|
| 1 | .30 | .10 |
| 2 | .62 | .33 |
| 3 | 1.16 | .58 |
| 4 | 1.30 | .86 |
| 5 | 1.61 | 1.20 |
| 6 | 1.64 | 1.60 |
| 7 | 1.71 | 2.10 |
| 8 | 1.79 | 2.77 |
| 9 | 3.53 | 3.79 |
| 10 | 4.38 | 5.99 |

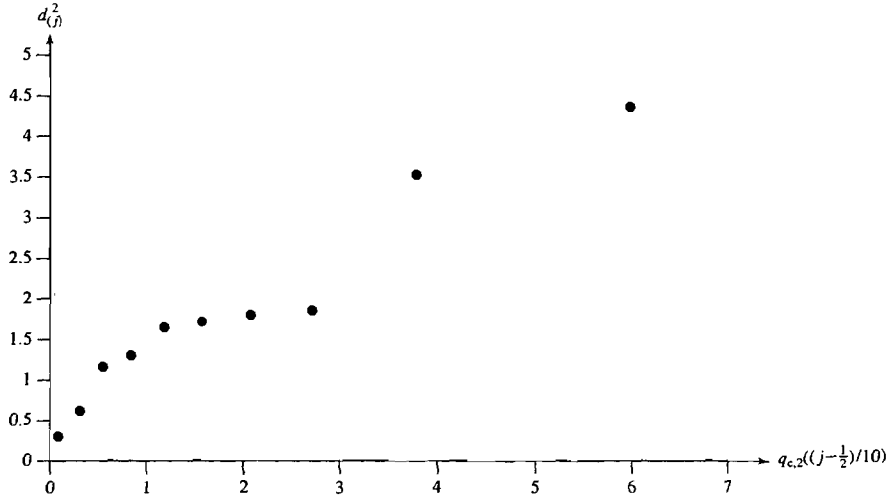



**Figure 4.7** A chi-square plot of the ordered distances in Example 4.13.

A graph of the pairs $\left(q_{c,2}\left(\left(j-\frac{1}{2}\right)/10\right), d^2_{(j)}\right)$ is shown in Figure 4.7. The points in Figure 4.7 are reasonably straight. Given the small sample size it is difficult to reject bivariate normality on the evidence in this graph. If further analysis of the data were required, it might be reasonable to transform them to observations more nearly bivariate normal. Appropriate transformations are discussed in Section 4.8. ■

In addition to inspecting univariate plots and scatter plots, we should check multivariate normality by constructing a chi-squared or $d^2$ plot. Figure 4.8 contains $d^2$

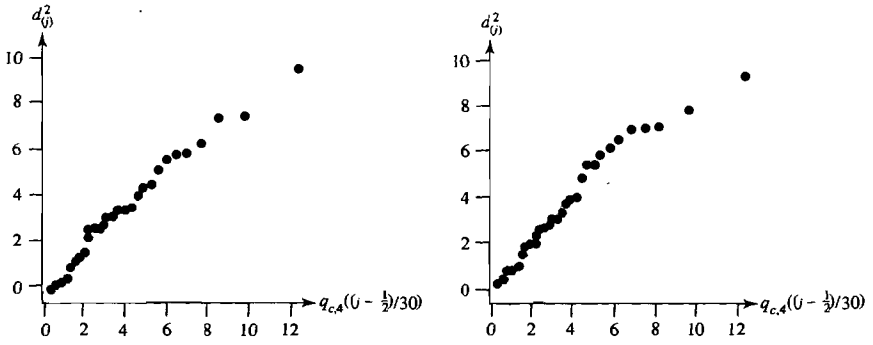



**Figure 4.8** Chi-square plots for two simulated four-variate normal data sets with $n = 30$.

plots based on two computer-generated samples of 30 four-variate normal random vectors. As expected, the plots have a straight-line pattern, but the top two or three ordered squared distances are quite variable.

The next example contains a real data set comparable to the simulated data set that produced the plots in Figure 4.8.

---

**Example 4.14 (Evaluating multivariate normality for a four-variable data set)** The data in Table 4.3 were obtained by taking four different measures of stiffness, $x_1$, $x_2$, $x_3$, and $x_4$, of each of $n = 30$ boards. The first measurement involves sending a shock wave down the board, the second measurement is determined while vibrating the board, and the last two measurements are obtained from static tests. The squared distances $d_j^2 = (\mathbf{x}_j - \bar{\mathbf{x}})'\mathbf{S}^{-1}(\mathbf{x}_j - \bar{\mathbf{x}})$ are also presented in the table.

**Table 4.3** Four Measurements of Stiffness

| Observation no. | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $d^2$ | Observation no. | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $d^2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1889 | 1651 | 1561 | 1778 | .60 | 16 | 1954 | 2149 | 1180 | 1281 | 16.85 |
| 2 | 2403 | 2048 | 2087 | 2197 | 5.48 | 17 | 1325 | 1170 | 1002 | 1176 | 3.50 |
| 3 | 2119 | 1700 | 1815 | 2222 | 7.62 | 18 | 1419 | 1371 | 1252 | 1308 | 3.99 |
| 4 | 1645 | 1627 | 1110 | 1533 | 5.21 | 19 | 1828 | 1634 | 1602 | 1755 | 1.36 |
| 5 | 1976 | 1916 | 1614 | 1883 | 1.40 | 20 | 1725 | 1594 | 1313 | 1646 | 1.46 |
| 6 | 1712 | 1712 | 1439 | 1546 | 2.22 | 21 | 2276 | 2189 | 1547 | 2111 | 9.90 |
| 7 | 1943 | 1685 | 1271 | 1671 | 4.99 | 22 | 1899 | 1614 | 1422 | 1477 | 5.06 |
| 8 | 2104 | 1820 | 1717 | 1874 | 1.49 | 23 | 1633 | 1513 | 1290 | 1516 | .80 |
| 9 | 2983 | 2794 | 2412 | 2581 | 12.26 | 24 | 2061 | 1867 | 1646 | 2037 | 2.54 |
| 10 | 1745 | 1600 | 1384 | 1508 | .77 | 25 | 1856 | 1493 | 1356 | 1533 | 4.58 |
| 11 | 1710 | 1591 | 1518 | 1667 | 1.93 | 26 | 1727 | 1412 | 1238 | 1469 | 3.40 |
| 12 | 2046 | 1907 | 1627 | 1898 | .46 | 27 | 2168 | 1896 | 1701 | 1834 | 2.38 |
| 13 | 1840 | 1841 | 1595 | 1741 | 2.70 | 28 | 1655 | 1675 | 1414 | 1597 | 3.00 |
| 14 | 1867 | 1685 | 1493 | 1678 | .13 | 29 | 2326 | 2301 | 2065 | 2234 | 6.28 |
| 15 | 1859 | 1649 | 1389 | 1714 | 1.08 | 30 | 1490 | 1382 | 1214 | 1284 | 2.58 |

Source: Data courtesy of William Galligan.

The marginal distributions appear quite normal (see Exercise 4.33), with the possible exception of specimen (board) 9.

To further evaluate multivariate normality, we constructed the chi-square plot shown in Figure 4.9. The two specimens with the largest squared distances are clearly removed from the straight-line pattern. Together, with the next largest point or two, they make the plot appear curved at the upper end. We will return to a discussion of this plot in Example 4.15. ■

We have discussed some rather simple techniques for checking the multivariate normality assumption. Specifically, we advocate calculating the $d_j^2$, $j = 1, 2, \ldots, n$ [see Equation (4-32)] and comparing the results with $\chi^2$ quantiles. For example, $p$-variate normality is indicated if

1. Roughly half of the $d_j^2$ are less than or equal to $q_{c,p}(.50)$.

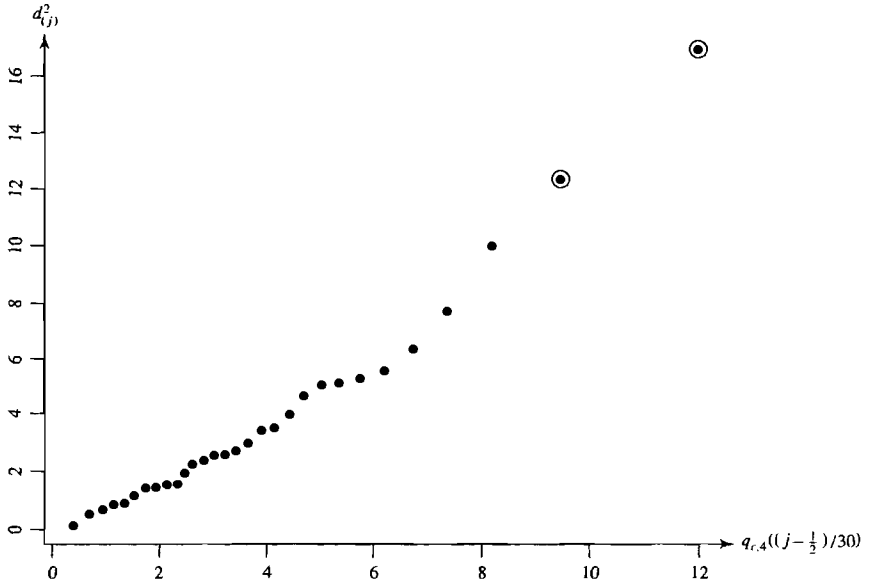



**Figure 4.9** A chi-square plot for the data in Example 4.14.

2. A plot of the ordered squared distances $d^2_{(1)} \leq d^2_{(2)} \leq \cdots \leq d^2_{(n)}$ versus $q_{c,p}\left(\frac{1-\frac{1}{2}}{n}\right), q_{c,p}\left(\frac{2-\frac{1}{2}}{n}\right), \ldots, q_{c,p}\left(\frac{n-\frac{1}{2}}{n}\right)$, respectively, is nearly a straight line having slope 1 and that passes through the origin.

(See [6] for a more complete exposition of methods for assessing normality.)

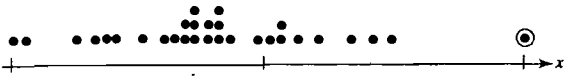
We close this section by noting that all measures of goodness of fit suffer the same serious drawback. When the sample size is small, only the most aberrant behavior will be identified as lack of fit. On the other hand, very large samples invariably produce statistically significant lack of fit. Yet the departure from the specified distribution may be very small and technically unimportant to the inferential conclusions.

## 4.7 Detecting Outliers and Cleaning Data

Most data sets contain one or a few unusual observations that do not seem to belong to the pattern of variability produced by the other observations. With data on a single characteristic, unusual observations are those that are either very large or very small relative to the others. The situation can be more complicated with multivariate data. Before we address the issue of identifying these *outliers*, we must emphasize that not all outliers are wrong numbers. They may, justifiably, be part of the group and may lead to a better understanding of the phenomena being studied.

Outliers are best detected visually whenever this is possible. When the number of observations $n$ is large, dot plots are not feasible. When the number of characteristics $p$ is large, the large number of scatter plots $p(p-1)/2$ may prevent viewing them all. Even so, we suggest first visually inspecting the data whenever possible.

What should we look for? For a single random variable, the problem is one dimensional, and we look for observations that are far from the others. For instance, the dot diagram




reveals a single large observation which is circled.

In the bivariate case, the situation is more complicated. Figure 4.10 shows a situation with two unusual observations.

The data point circled in the upper right corner of the figure is detached from the pattern, and its second coordinate is large relative to the rest of the $x_2$

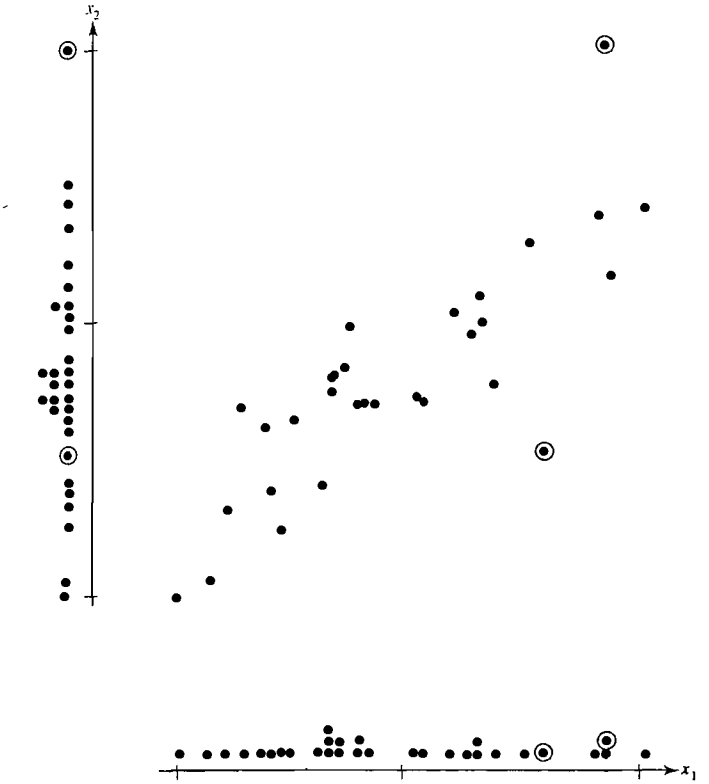



**Figure 4.10** Two outliers; one univariate and one bivariate.

measurements, as shown by the vertical dot diagram. The second outlier, also circled, is far from the elliptical pattern of the rest of the points, but, separately, each of its components has a typical value. This outlier cannot be detected by inspecting the marginal dot diagrams.

In higher dimensions, there can be outliers that cannot be detected from the univariate plots or even the bivariate scatter plots. Here a large value of $(\mathbf{x}_j - \bar{\mathbf{x}})'\mathbf{S}^{-1}(\mathbf{x}_j - \bar{\mathbf{x}})$ will suggest an unusual observation, even though it cannot be seen visually.

## Steps for Detecting Outliers

1. Make a dot plot for each variable.
2. Make a scatter plot for each pair of variables.
3. Calculate the standardized values $z_{jk} = (x_{jk} - \bar{x}_k)/\sqrt{s_{kk}}$ for $j = 1, 2, \ldots, n$ and each column $k = 1, 2, \ldots, p$. Examine these standardized values for large or small values.
4. Calculate the generalized squared distances $(\mathbf{x}_j - \bar{\mathbf{x}})'\mathbf{S}^{-1}(\mathbf{x}_j - \bar{\mathbf{x}})$. Examine these distances for unusually large values. In a chi-square plot, these would be the points farthest from the origin.

In step 3, "large" must be interpreted relative to the sample size and number of variables. There are $n \times p$ standardized values. When $n = 100$ and $p = 5$, there are 500 values. You expect 1 or 2 of these to exceed 3 or be less than $-3$, even if the data came from a multivariate distribution that is exactly normal. As a guideline, 3.5 might be considered large for moderate sample sizes.

In step 4, "large" is measured by an appropriate percentile of the chi-square distribution with $p$ degrees of freedom. If the sample size is $n = 100$, we would expect 5 observations to have values of $d_j^2$ that exceed the upper fifth percentile of the chi-square distribution. A more extreme percentile must serve to determine observations that do not fit the pattern of the remaining data.

The data we presented in Table 4.3 concerning lumber have already been cleaned up somewhat. Similar data sets from the same study also contained data on $x_5$ = tensile strength. Nine observation vectors, out of the total of 112, are given as rows in the following table, along with their standardized values.

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $z_1$ | $z_2$ | $z_3$ | $z_4$ | $z_5$ |
|---|---|---|---|---|---|---|---|---|---|
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | |
| 1631 | 1528 | 1452 | 1559 | 1602 | .06 | −.15 | .05 | .28 | −.12 |
| 1770 | 1677 | 1707 | 1738 | 1785 | .64 | .43 | 1.07 | .94 | .60 |
| 1376 | 1190 | 723 | 1285 | 2791 | −1.01 | −1.47 | −2.87 | −.73 | (4.57) |
| 1705 | 1577 | 1332 | 1703 | 1664 | .37 | .04 | −.43 | .81 | .13 |
| 1643 | 1535 | 1510 | 1494 | 1582 | .11 | −.12 | .28 | .04 | −.20 |
| 1567 | 1510 | 1301 | 1405 | 1553 | −.21 | −.22 | −.56 | −.28 | −.31 |
| 1528 | 1591 | 1714 | 1685 | 1698 | −.38 | .10 | 1.10 | .75 | .26 |
| 1803 | 1826 | 1748 | 2746 | 1764 | .78 | 1.01 | 1.23 | (4.65) | .52 |
| 1587 | 1554 | 1352 | 1554 | 1551 | −.13 | −.05 | −.35 | .26 | −.32 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | |

The standardized values are based on the sample mean and variance, calculated from all 112 observations. There are two extreme standardized values. Both are too large with standardized values over 4.5. During their investigation, the researchers recorded measurements by hand in a logbook and then performed calculations that produced the values given in the table. When they checked their records regarding the values pinpointed by this analysis, errors were discovered. The value $x_5 = 2791$ was corrected to 1241, and $x_4 = 2746$ was corrected to 1670. Incorrect readings on an individual variable are quickly detected by locating a large leading digit for the standardized value.

The next example returns to the data on lumber discussed in Example 4.14.

**Example 4.15 (Detecting outliers in the data on lumber)** Table 4.4 contains the data in Table 4.3, along with the standardized observations. These data consist of four different measures of stiffness $x_1, x_2, x_3$, and $x_4$, on each of $n = 30$ boards. Recall that the first measurement involves sending a shock wave down the board, the second measurement is determined while vibrating the board, and the last two measurements are obtained from static tests. The standardized measurements are

**Table 4.4** Four Measurements of Stiffness with Standardized Values

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | Observation no. | $z_1$ | $z_2$ | $z_3$ | $z_4$ | $d^2$ |
|---|---|---|---|---|---|---|---|---|---|
| 1889 | 1651 | 1561 | 1778 | 1 | −.1 | −.3 | .2 | .2 | .60 |
| 2403 | 2048 | 2087 | 2197 | 2 | 1.5 | .9 | 1.9 | 1.5 | 5.48 |
| 2119 | 1700 | 1815 | 2222 | 3 | .7 | −.2 | 1.0 | 1.5 | 7.62 |
| 1645 | 1627 | 1110 | 1533 | 4 | −.8 | −.4 | −1.3 | −.6 | 5.21 |
| 1976 | 1916 | 1614 | 1883 | 5 | .2 | .5 | .3 | .5 | 1.40 |
| 1712 | 1712 | 1439 | 1546 | 6 | −.6 | −.1 | −.2 | −.6 | 2.22 |
| 1943 | 1685 | 1271 | 1671 | 7 | .1 | −.2 | −.8 | −.2 | 4.99 |
| 2104 | 1820 | 1717 | 1874 | 8 | .6 | .2 | .7 | .5 | 1.49 |
| 2983 | 2794 | 2412 | 2581 | 9 | 3.3 | 3.3 | 3.0 | 2.7 | 12.26 |
| 1745 | 1600 | 1384 | 1508 | 10 | −.5 | −.5 | −.4 | −.7 | .77 |
| 1710 | 1591 | 1518 | 1667 | 11 | −.6 | −.5 | .0 | −.2 | 1.93 |
| 2046 | 1907 | 1627 | 1898 | 12 | .4 | .5 | .4 | .5 | .46 |
| 1840 | 1841 | 1595 | 1741 | 13 | −.2 | .3 | .3 | .0 | 2.70 |
| 1867 | 1685 | 1493 | 1678 | 14 | −.1 | −.2 | −.1 | −.1 | .13 |
| 1859 | 1649 | 1389 | 1714 | 15 | −.1 | −.3 | −.4 | −.0 | 1.08 |
| 1954 | 2149 | 1180 | 1281 | 16 | .1 | 1.3 | −1.1 | −1.4 | 16.85 |
| 1325 | 1170 | 1002 | 1176 | 17 | −1.8 | −1.8 | −1.7 | −1.7 | 3.50 |
| 1419 | 1371 | 1252 | 1308 | 18 | −1.5 | −1.2 | −.8 | −1.3 | 3.99 |
| 1828 | 1634 | 1602 | 1755 | 19 | −.2 | −.4 | .3 | .1 | 1.36 |
| 1725 | 1594 | 1313 | 1646 | 20 | −.6 | −.5 | −.6 | −.2 | 1.46 |
| 2276 | 2189 | 1547 | 2111 | 21 | 1.1 | 1.4 | .1 | 1.2 | 9.90 |
| 1899 | 1614 | 1422 | 1477 | 22 | −.0 | −.4 | −.3 | −.8 | 5.06 |
| 1633 | 1513 | 1290 | 1516 | 23 | −.8 | −.7 | −.7 | −.6 | .80 |
| 2061 | 1867 | 1646 | 2037 | 24 | .5 | .4 | .5 | 1.0 | 2.54 |
| 1856 | 1493 | 1356 | 1533 | 25 | −.2 | −.8 | −.5 | −.6 | 4.58 |
| 1727 | 1412 | 1238 | 1469 | 26 | −.6 | −1.1 | −.9 | −.8 | 3.40 |
| 2168 | 1896 | 1701 | 1834 | 27 | .8 | .5 | .6 | .3 | 2.38 |
| 1655 | 1675 | 1414 | 1597 | 28 | −.8 | −.2 | −.3 | −.4 | 3.00 |
| 2326 | 2301 | 2065 | 2234 | 29 | 1.3 | 1.7 | 1.8 | 1.6 | 6.28 |
| 1490 | 1382 | 1214 | 1284 | 30 | −1.3 | −1.2 | −1.0 | −1.4 | 2.58 |

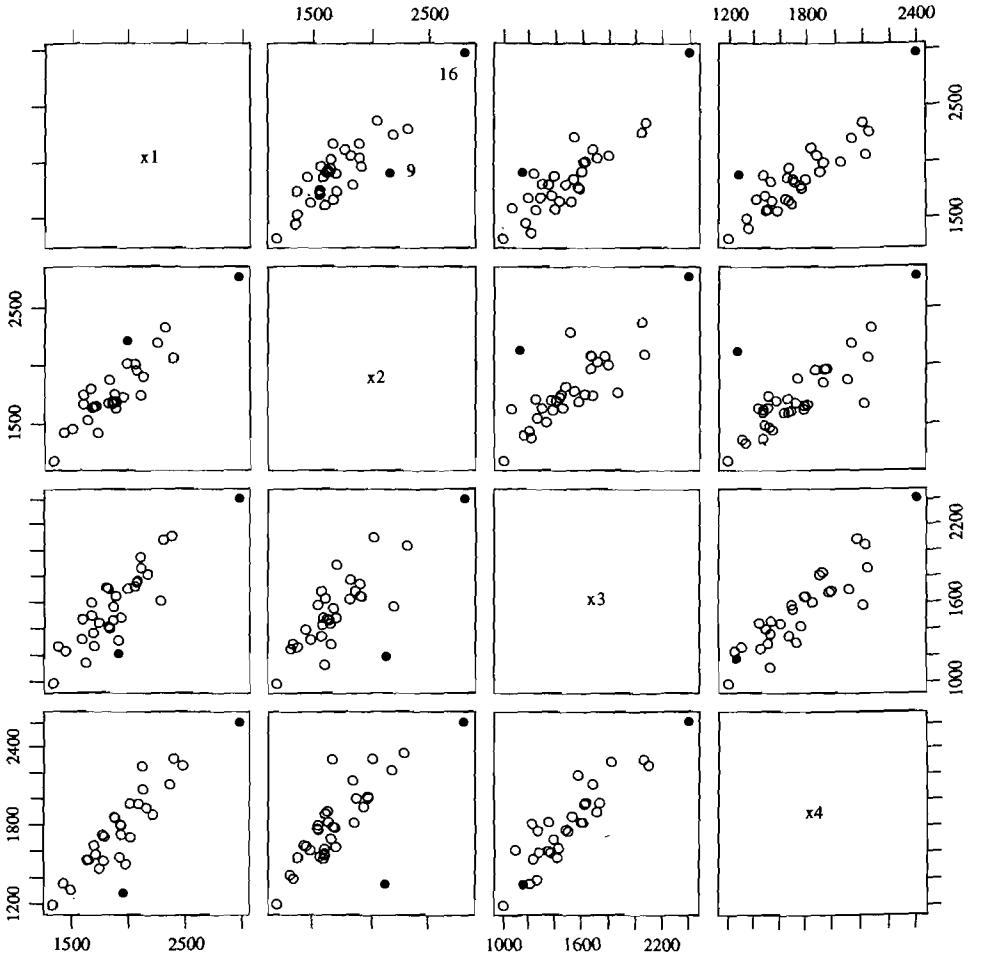



**Figure 4.11** Scatter plots for the lumber stiffness data with specimens 9 and 16 plotted as solid dots.

$$z_{jk} = \frac{x_{jk} - \bar{x}_k}{\sqrt{s_{kk}}}, \qquad k = 1, 2, 3, 4; \qquad j = 1, 2, \ldots, 30$$

and the squares of the distances are $d_j^2 = (\mathbf{x}_j - \bar{\mathbf{x}})' \mathbf{S}^{-1} (\mathbf{x}_j - \bar{\mathbf{x}})$.

The last column in Table 4.4 reveals that specimen 16 is a multivariate outlier, since $\chi_4^2(.005) = 14.86$; yet all of the individual measurements are well within their respective univariate scatters. Specimen 9 also has a large $d^2$ value.

The two specimens (9 and 16) with large squared distances stand out as clearly different from the rest of the pattern in Figure 4.9. Once these two points are removed, the remaining pattern conforms to the expected straight-line relation. Scatter plots for the lumber stiffness measurements are given in Figure 4.11 above.

The solid dots in these figures correspond to specimens 9 and 16. Although the dot for specimen 16 stands out in all the plots, the dot for specimen 9 is "hidden" in the scatter plot of $x_3$ versus $x_4$ and nearly hidden in that of $x_1$ versus $x_3$. However, specimen 9 is clearly identified as a multivariate outlier when all four variables are considered.

Scientists specializing in the properties of wood conjectured that specimen 9 was unusually clear and therefore very stiff and strong. It would also appear that specimen 16 is a bit unusual, since both of its dynamic measurements are above average and the two static measurements are low. Unfortunately, it was not possible to investigate this specimen further because the material was no longer available. ■

If outliers are identified, they should be examined for content, as was done in the case of the data on lumber stiffness in Example 4.15. Depending upon the nature of the outliers and the objectives of the investigation, outliers may be deleted or appropriately "weighted" in a subsequent analysis.

Even though many statistical techniques assume normal populations, those based on the sample mean vectors usually will not be disturbed by a few moderate outliers. Hawkins [7] gives an extensive treatment of the subject of outliers.

## 4.8 Transformations to Near Normality

If normality is not a viable assumption, what is the next step? One alternative is to ignore the findings of a normality check and proceed as if the data were normally distributed. This practice is not recommended, since, in many instances, it could lead to incorrect conclusions. A second alternative is to make nonnormal data more "normal looking" by considering *transformations* of the data. Normal-theory analyses can then be carried out with the suitably transformed data.

Transformations are nothing more than a reexpression of the data in different units. For example, when a histogram of positive observations exhibits a long right-hand tail, transforming the observations by taking their logarithms or square roots will often markedly improve the symmetry about the mean and the approximation to a normal distribution. It frequently happens that the new units provide more natural expressions of the characteristics being studied.

Appropriate transformations are suggested by (1) theoretical considerations or (2) the data themselves (or both). It has been shown theoretically that data that are counts can often be made more normal by taking their *square roots*. Similarly, the *logit transformation* applied to proportions and *Fisher's z-transformation* applied to correlation coefficients yield quantities that are approximately normally distributed.

### Helpful Transformations To Near Normality

| *Original Scale* | *Transformed Scale* | |
|---|---|---|
| 1. Counts, $y$ | $\sqrt{y}$ | |
| 2. Proportions, $\hat{p}$ | $\text{logit}(\hat{p}) = \frac{1}{2}\log\left(\frac{\hat{p}}{1-\hat{p}}\right)$ | (4-33) |
| 3. Correlations, $r$ | Fisher's $z(r) = \frac{1}{2}\log\left(\frac{1+r}{1-r}\right)$ | |

In many instances, the choice of a transformation to improve the approximation to normality is not obvious. For such cases, it is convenient to let the data suggest a transformation. A useful family of transformations for this purpose is the family of *power transformations*.

Power transformations are defined only for positive variables. However, this is not as restrictive as it seems, because a single constant can be added to each observation in the data set if some of the values are negative.

Let $x$ represent an arbitrary observation. The power family of transformations is indexed by a parameter $\lambda$. A given value for $\lambda$ implies a particular transformation. For example, consider $x^\lambda$ with $\lambda = -1$. Since $x^{-1} = 1/x$, this choice of $\lambda$ corresponds to the reciprocal transformation. We can trace the family of transformations as $\lambda$ ranges from negative to positive powers of $x$. For $\lambda = 0$, we define $x^0 = \ln x$. A sequence of possible transformations is

$$\underbrace{\ldots, x^{-1} = \frac{1}{x}, x^0 = \ln x, x^{1/4} = \sqrt[4]{x}, x^{1/2} = \sqrt{x},}_{\text{shrinks large values of } x} \qquad \underbrace{x^2, x^3, \ldots}_{\text{increases large values of } x}$$

To select a power transformation, an investigator looks at the marginal dot diagram or histogram and decides whether large values have to be "pulled in" or "pushed out" to improve the symmetry about the mean. Trial-and-error calculations with a few of the foregoing transformations should produce an improvement. The final choice should always be examined by a $Q$–$Q$ plot or other checks to see whether the tentative normal assumption is satisfactory.

The transformations we have been discussing are data based in the sense that it is only the appearance of the data themselves that influences the choice of an appropriate transformation. There are no external considerations involved, although the transformation actually used is often determined by some mix of information supplied by the data and extra-data factors, such as simplicity or ease of interpretation.

A convenient analytical method is available for choosing a power transformation. We begin by focusing our attention on the univariate case.

Box and Cox [3] consider the slightly modified family of power transformations

$$x^{(\lambda)} = \begin{cases} \dfrac{x^\lambda - 1}{\lambda} & \lambda \neq 0 \\ \ln x & \lambda = 0 \end{cases} \tag{4-34}$$

which is continuous in $\lambda$ for $x > 0$. (See [8].) Given the observations $x_1, x_2, \ldots, x_n$, the Box–Cox solution for the choice of an appropriate power $\lambda$ is the solution that *maximizes* the expression

$$\ell(\lambda) = -\frac{n}{2} \ln \left[ \frac{1}{n} \sum_{j=1}^{n} (x_j^{(\lambda)} - \overline{x^{(\lambda)}})^2 \right] + (\lambda - 1) \sum_{j=1}^{n} \ln x_j \tag{4-35}$$

We note that $x_j^{(\lambda)}$ is defined in (4-34) and

$$\overline{x^{(\lambda)}} = \frac{1}{n} \sum_{j=1}^{n} x_j^{(\lambda)} = \frac{1}{n} \sum_{j=1}^{n} \left( \frac{x_j^\lambda - 1}{\lambda} \right) \tag{4-36}$$

is the arithmetic average of the transformed observations. The first term in (4-35) is, apart from a constant, the logarithm of a normal likelihood function, after maximizing it with respect to the population mean and variance parameters.

The calculation of $\ell(\lambda)$ for many values of $\lambda$ is an easy task for a computer. It is helpful to have a graph of $\ell(\lambda)$ versus $\lambda$, as well as a tabular display of the pairs $(\lambda, \ell(\lambda))$, in order to study the behavior near the maximizing value $\hat{\lambda}$. For instance, if either $\lambda = 0$ (logarithm) or $\lambda = \frac{1}{2}$ (square root) is near $\hat{\lambda}$, one of these may be preferred because of its simplicity.

Rather than program the calculation of (4-35), some statisticians recommend the equivalent procedure of fixing $\lambda$, creating the new variable

$$y_j^{(\lambda)} = \frac{x_j^{\lambda} - 1}{\lambda\left[\left(\prod_{i=1}^{n} x_i\right)^{1/n}\right]^{\lambda-1}} \qquad j = 1, \ldots, n \tag{4-37}$$

and then calculating the sample variance. The minimum of the variance occurs at the same $\lambda$ that maximizes (4-35).

*Comment.* It is now understood that the transformation obtained by maximizing $\ell(\lambda)$ usually improves the approximation to normality. However, there is no guarantee that even the best choice of $\lambda$ will produce a transformed set of values that adequately conform to a normal distribution. The outcomes produced by a transformation selected according to (4-35) should always be carefully examined for possible violations of the tentative assumption of normality. This warning applies with equal force to transformations selected by any other technique.

**Example 4.16 (Determining a power transformation for univariate data)** We gave readings of the microwave radiation emitted through the closed doors of $n = 42$ ovens in Example 4.10. The $Q$–$Q$ plot of these data in Figure 4.6 indicates that the observations deviate from what would be expected if they were normally distributed. Since all the observations are positive, let us perform a power transformation of the data which, we hope, will produce results that are more nearly normal. Restricting our attention to the family of transformations in (4-34), we must find that value of $\lambda$ maximizing the function $\ell(\lambda)$ in (4-35).

The pairs $(\lambda, \ell(\lambda))$ are listed in the following table for several values of $\lambda$:

| $\lambda$ | $\ell(\lambda)$ | $\lambda$ | $\ell(\lambda)$ |
|---|---|---|---|
| −1.00 | 70.52 | | |
| −.90 | 75.65 | .40 | 106.20 |
| −.80 | 80.46 | .50 | 105.50 |
| −.70 | 84.94 | .60 | 104.43 |
| −.60 | 89.06 | .70 | 103.03 |
| −.50 | 92.79 | .80 | 101.33 |
| −.40 | 96.10 | .90 | 99.34 |
| −.30 | 98.97 | 1.00 | 97.10 |
| −.20 | 101.39 | 1.10 | 94.64 |
| −.10 | 103.35 | 1.20 | 91.96 |
| .00 | 104.83 | 1.30 | 89.10 |
| .10 | 105.84 | 1.40 | 86.07 |
| .20 | 106.39 | 1.50 | 82.88 |
| .30 | 106.51 | | |

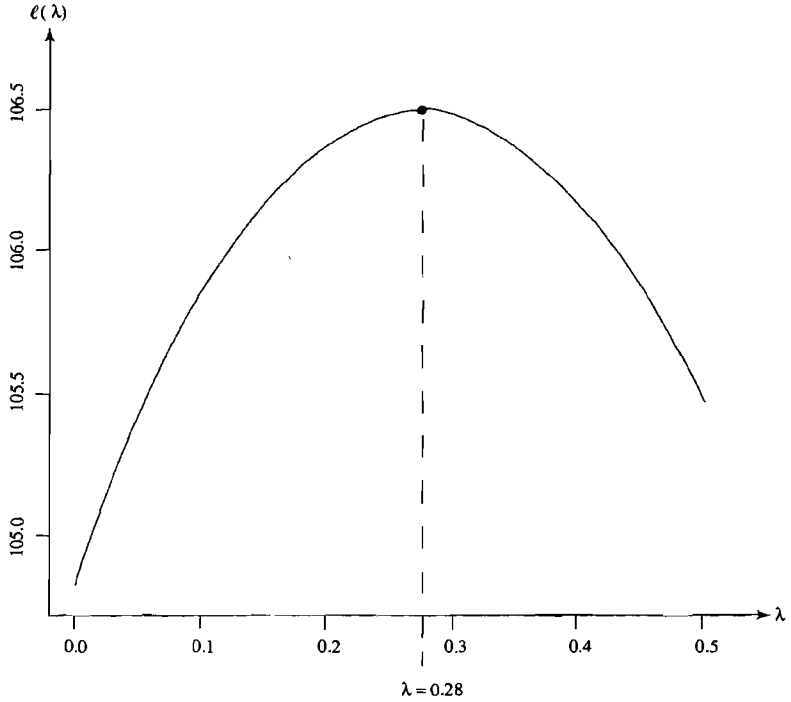



**Figure 4.12** Plot of $\ell(\lambda)$ versus $\lambda$ for radiation data (door closed).

The curve of $\ell(\lambda)$ versus $\lambda$ that allows the more exact determination $\hat{\lambda} = .28$ is shown in Figure 4.12 .

It is evident from both the table and the plot that a value of $\hat{\lambda}$ around .30 maximizes $\ell(\lambda)$. For convenience, we choose $\hat{\lambda} = .25$. The data $x_j$ were reexpressed as

$$x_j^{(1/4)} = \frac{x_j^{1/4} - 1}{\frac{1}{4}} \qquad j = 1, 2, \ldots, 42$$

and a $Q$–$Q$ plot was constructed from the transformed quantities. This plot is shown in Figure 4.13 on page 196. The quantile pairs fall very close to a straight line, and we would conclude from this evidence that the $x_j^{(1/4)}$ are approximately normal. ■

## Transforming Multivariate Observations

With multivariate observations, a power transformation must be selected for each of the variables. Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ be the power transformations for the $p$ measured characteristics. Each $\lambda_k$ can be selected by *maximizing*

$$\ell_k(\lambda) = -\frac{n}{2} \ln \left[ \frac{1}{n} \sum_{j=1}^{n} (x_{jk}^{(\lambda_k)} - \overline{x_k^{(\lambda_k)}})^2 \right] + (\lambda_k - 1) \sum_{j=1}^{n} \ln x_{jk} \qquad (4\text{-}38)$$

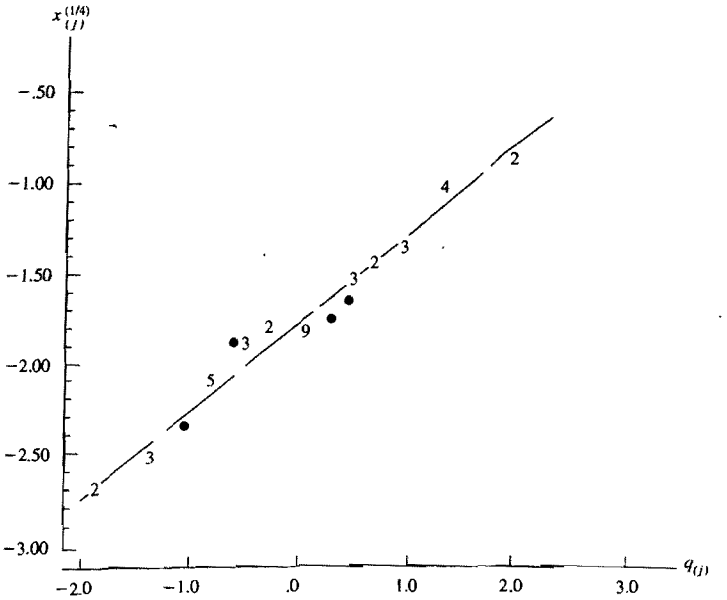



**Figure 4.13** A $Q$–$Q$ plot of the transformed radiation data (door closed). (The integers in the plot indicate the number of points occupying the same location.)

where $x_{1k}, x_{2k}, \ldots, x_{nk}$ are the $n$ observations on the $k$th variable, $k = 1, 2, \ldots, p$. Here

$$\overline{x_k^{(\lambda_k)}} = \frac{1}{n}\sum_{j=1}^{n} x_{jk}^{(\lambda_k)} = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{x_{jk}^{\lambda_k} - 1}{\lambda_k}\right) \tag{4-39}$$

is the arithmetic average of the transformed observations. The $j$th transformed multivariate observation is

$$\mathbf{x}_j^{(\hat{\boldsymbol{\lambda}})} = \begin{bmatrix} \dfrac{x_{j1}^{\hat{\lambda}_1} - 1}{\hat{\lambda}_1} \\ \dfrac{x_{j2}^{\hat{\lambda}_2} - 1}{\hat{\lambda}_2} \\ \vdots \\ \dfrac{x_{jp}^{\hat{\lambda}_p} - 1}{\hat{\lambda}_p} \end{bmatrix}$$

where $\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_p$ are the values that individually maximize (4-38).

The procedure just described is equivalent to making each marginal distribution approximately normal. Although normal marginals are not sufficient to ensure that the joint distribution is normal, in practical applications this may be good enough. If not, we could start with the values $\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_p$ obtained from the preceding transformations and iterate toward the set of values $\boldsymbol{\lambda}' = [\lambda_1, \lambda_2, \ldots, \lambda_p]$, which collectively maximizes

$$\ell(\lambda_1, \lambda_2, \ldots, \lambda_p) = -\frac{n}{2}\ln|\mathbf{S}(\boldsymbol{\lambda})| + (\lambda_1 - 1)\sum_{j=1}^{n}\ln x_{j1} + (\lambda_2 - 1)\sum_{j=1}^{n}\ln x_{j2} + \cdots + (\lambda_p - 1)\sum_{j=1}^{n}\ln x_{jp} \tag{4-40}$$

where $\mathbf{S}(\boldsymbol{\lambda})$ is the sample covariance matrix computed from

$$\mathbf{x}_j^{(\boldsymbol{\lambda})} = \begin{bmatrix} \dfrac{x_{j1}^{\lambda_1} - 1}{\lambda_1} \\ \dfrac{x_{j2}^{\lambda_2} - 1}{\lambda_2} \\ \vdots \\ \dfrac{x_{jp}^{\lambda_p} - 1}{\lambda_p} \end{bmatrix} \qquad j = 1, 2, \ldots, n$$

Maximizing (4-40) not only is substantially more difficult than maximizing the individual expressions in (4-38), but also is unlikely to yield remarkably better results. The selection method based on Equation (4-40) is equivalent to maximizing a multivariate likelihood over $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$ and $\boldsymbol{\lambda}$, whereas the method based on (4-38) corresponds to maximizing the $k$th univariate likelihood over $\mu_k$, $\sigma_{kk}$, and $\lambda_k$. The latter likelihood is generated by pretending there is some $\lambda_k$ for which the observations $(x_{jk}^{\lambda_k} - 1)/\lambda_k$, $j = 1, 2, \ldots, n$ have a normal distribution. See [3] and [2] for detailed discussions of the univariate and multivariate cases, respectively. (Also, see [8].)

---

**Example 4.17 (Determining power transformations for bivariate data)** Radiation measurements were also recorded through the open doors of the $n = 42$ microwave ovens introduced in Example 4.10. The amount of radiation emitted through the open doors of these ovens is listed in Table 4.5.

In accordance with the procedure outlined in Example 4.16, a power transformation for these data was selected by maximizing $\ell(\lambda)$ in (4-35). The approximate maximizing value was $\hat{\lambda} = .30$. Figure 4.14 on page 199 shows $Q$–$Q$ plots of the untransformed and transformed door-open radiation data. (These data were actually

**Table 4.5** Radiation Data (Door Open)

| Oven no. | Radiation | Oven no. | Radiation | Oven no. | Radiation |
|---|---|---|---|---|---|
| 1 | .30 | 16 | .20 | 31 | .10 |
| 2 | .09 | 17 | .04 | 32 | .10 |
| 3 | .30 | 18 | .10 | 33 | .10 |
| 4 | .10 | 19 | .01 | 34 | .30 |
| 5 | .10 | 20 | .60 | 35 | .12 |
| 6 | .12 | 21 | .12 | 36 | .25 |
| 7 | .09 | 22 | .10 | 37 | .20 |
| 8 | .10 | 23 | .05 | 38 | .40 |
| 9 | .09 | 24 | .05 | 39 | .33 |
| 10 | .10 | 25 | .15 | 40 | .32 |
| 11 | .07 | 26 | .30 | 41 | .12 |
| 12 | .05 | 27 | .15 | 42 | .12 |
| 13 | .01 | 28 | .09 | | |
| 14 | .45 | 29 | .09 | | |
| 15 | .12 | 30 | .28 | | |

Source: Data courtesy of J. D. Cryer.

transformed by taking the fourth root, as in Example 4.16.) It is clear from the figure that the transformed data are more nearly normal, although the normal approximation is not as good as it was for the door-closed data.

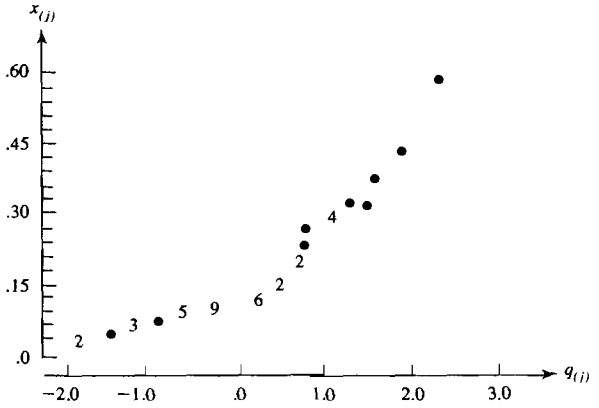
Let us denote the door-closed data by $x_{11}, x_{21}, \ldots, x_{42,1}$ and the door-open data by $x_{12}, x_{22}, \ldots, x_{42,2}$. Choosing a power transformation for each set by maximizing the expression in (4-35) is equivalent to maximizing $\ell_k(\lambda)$ in (4-38) with $k = 1, 2$. Thus, using the outcomes from Example 4.16 and the foregoing results, we have $\hat{\lambda}_1 = .30$ and $\hat{\lambda}_2 = .30$. These powers were determined for the *marginal* distributions of $x_1$ and $x_2$.

We can consider the *joint* distribution of $x_1$ and $x_2$ and simultaneously determine the pair of powers $(\lambda_1, \lambda_2)$ that makes this joint distribution approximately bivariate normal. To do this, we must maximize $\ell(\lambda_1, \lambda_2)$ in (4-40) with respect to both $\lambda_1$ and $\lambda_2$.

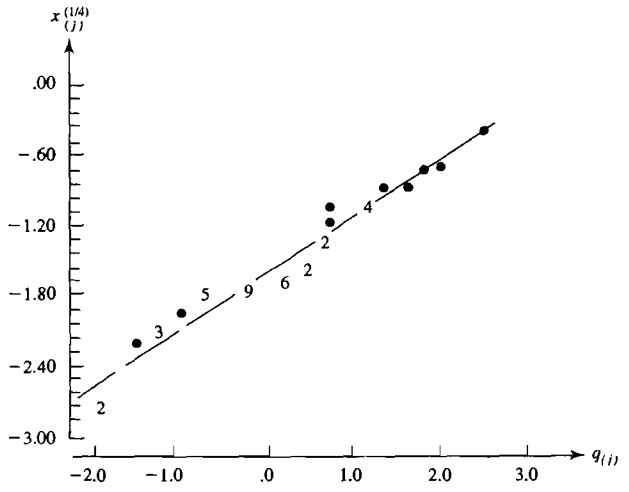
We computed $\ell(\lambda_1, \lambda_2)$ for a grid of $\lambda_1, \lambda_2$ values covering $0 \leq \lambda_1 \leq .50$ and $0 \leq \lambda_2 \leq .50$, and we constructed the contour plot shown in Figure 4.15 on page 200. We see that the maximum occurs at about $(\hat{\lambda}_1, \hat{\lambda}_2) = (.16, .16)$.

The "best" power transformations for this bivariate case do not differ substantially from those obtained by considering each marginal distribution. ■

As we saw in Example 4.17, making each marginal distribution approximately normal is roughly equivalent to addressing the bivariate distribution directly and making it approximately normal. It is generally easier to select appropriate transformations for the marginal distributions than for the joint distributions.




(a)




(b)

**Figure 4.14** *Q–Q* plots of (a) the original and (b) the transformed radiation data (with door open). (The integers in the plot indicate the number of points occupying the same location.)

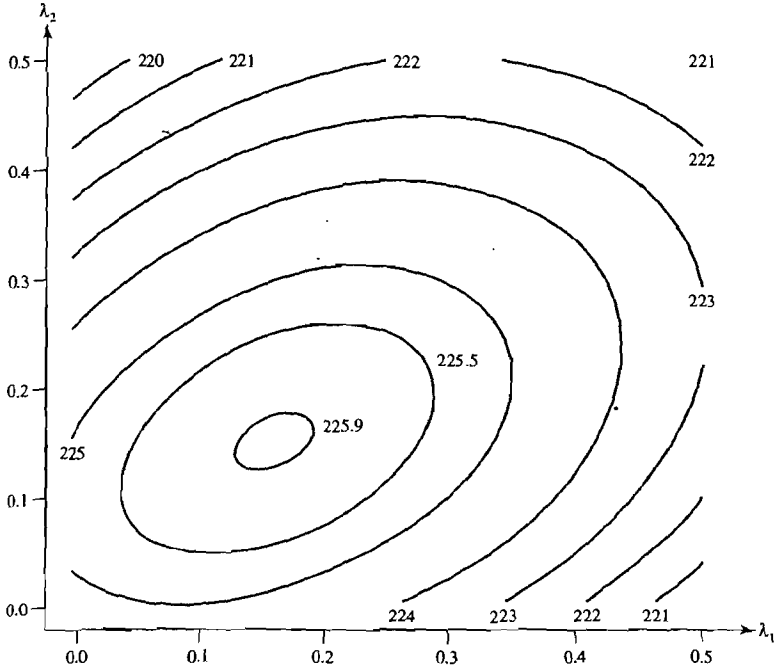



**Figure 4.15** Contour plot of $\ell(\lambda_1, \lambda_2)$ for the radiation data.

If the data includes some large negative values and have a single long tail, a more general transformation (see Yeo and Johnson [14]) should be applied.

$$x^{(\lambda)} = \begin{cases} \{(x+1)^{\lambda} - 1\}/\lambda & x \geq 0, \lambda \neq 0 \\ \ln(x+1) & x \geq 0, \lambda = 0 \\ -\{(-x+1)^{2-\lambda} - 1\}/(2-\lambda) & x < 0, \lambda \neq 2 \\ -\ln(-x+1) & x < 0, \lambda = 2 \end{cases}$$

## Exercises

**4.1.** Consider a bivariate normal distribution with $\mu_1 = 1$, $\mu_2 = 3$, $\sigma_{11} = 2$, $\sigma_{22} = 1$ and $\rho_{12} = -.8$.

(a) Write out the bivariate normal density.

(b) Write out the squared statistical distance expression $(\mathbf{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})$ as a quadratic function of $x_1$ and $x_2$.

**4.2.** Consider a bivariate normal population with $\mu_1 = 0$, $\mu_2 = 2$, $\sigma_{11} = 2$, $\sigma_{22} = 1$, and $\rho_{12} = .5$.

(a) Write out the bivariate normal density.

(b) Write out the squared generalized distance expression $(\mathbf{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})$ as a function of $x_1$ and $x_2$.

(c) Determine (and sketch) the constant-density contour that contains 50% of the probability.

**4.3.** Let $\mathbf{X}$ be $N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\mu}' = [-3, 1, 4]$ and

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Which of the following random variables are independent? Explain.

(a) $X_1$ and $X_2$

(b) $X_2$ and $X_3$

(c) $(X_1, X_2)$ and $X_3$

(d) $\dfrac{X_1 + X_2}{2}$ and $X_3$

(e) $X_2$ and $X_2 - \frac{5}{2}X_1 - X_3$

**4.4.** Let $\mathbf{X}$ be $N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\mu}' = [2, -3, 1]$ and

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

(a) Find the distribution of $3X_1 - 2X_2 + X_3$.

(b) Relabel the variables if necessary, and find a $2 \times 1$ vector $\mathbf{a}$ such that $X_2$ and $X_2 - \mathbf{a}'\begin{bmatrix} X_1 \\ X_3 \end{bmatrix}$ are independent.

**4.5.** Specify each of the following.

(a) The conditional distribution of $X_1$, given that $X_2 = x_2$ for the joint distribution in Exercise 4.2.

(b) The conditional distribution of $X_2$, given that $X_1 = x_1$ and $X_3 = x_3$ for the joint distribution in Exercise 4.3.

(c) The conditional distribution of $X_3$, given that $X_1 = x_1$ and $X_2 = x_2$ for the joint distribution in Exercise 4.4.

**4.6.** Let $\mathbf{X}$ be distributed as $N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu}' = [1, -1, 2]$ and

$$\boldsymbol{\Sigma} = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 5 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

Which of the following random variables are independent? Explain.

(a) $X_1$ and $X_2$

(b) $X_1$ and $X_3$

(c) $X_2$ and $X_3$

(d) $(X_1, X_3)$ and $X_2$

(e) $X_1$ and $X_1 + 3X_2 - 2X_3$

**4.7.** Refer to Exercise 4.6 and specify each of the following.

(a) The conditional distribution of $X_1$, given that $X_3 = x_3$.

(b) The conditional distribution of $X_1$, given that $X_2 = x_2$ and $X_3 = x_3$.

**4.8.** (Example of a nonnormal bivariate distribution with normal marginals.) Let $X_1$ be $N(0, 1)$, and let

$$X_2 = \begin{cases} -X_1 & \text{if } -1 \le X_1 \le 1 \\ X_1 & \text{otherwise} \end{cases}$$

Show each of the following.

(a) $X_2$ also has an $N(0, 1)$ distribution.

(b) $X_1$ and $X_2$ do *not* have a bivariate normal distribution.

*Hint:*

(a) Since $X_1$ is $N(0,1)$, $P[-1 < X_1 \le x] = P[-x \le X_1 < 1]$ for any $x$. When $-1 < x_2 < 1$, $P[X_2 \le x_2] = P[X_2 \le -1] + P[-1 < X_2 \le x_2] = P[X_1 \le -1] + P[-1 < -X_1 \le x_2] = P[X_1 \le -1] + P[-x_2 \le X_1 < 1]$. But $P[-x_2 \le X_1 < 1] = P[-1 < X_1 \le x_2]$ from the symmetry argument in the first line of this hint. Thus, $P[X_2 \le x_2] = P[X_1 \le -1] + P[-1 < X_1 \le x_2] = P[X_1 \le x_2]$, which is a standard normal probability.

(b) Consider the linear combination $X_1 - X_2$, which equals zero with probability $P[|X_1| > 1] = .3174$.

**4.9.** Refer to Exercise 4.8, but modify the construction by replacing the break point 1 by $c$ so that

$$X_2 = \begin{cases} -X_1 & \text{if } -c \le X_1 \le c \\ X_1 & \text{elsewhere} \end{cases}$$

Show that $c$ can be chosen so that $\text{Cov}(X_1, X_2) = 0$, but that the two random variables are not independent.

*Hint:*
For $c = 0$, evaluate $\text{Cov}(X_1, X_2) = E[X_1(X_1)]$
For $c$ very large, evaluate $\text{Cov}(X_1, X_2) \doteq E[X_1(-X_1)]$.

**4.10.** Show each of the following.

(a)

$$\begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} = |\mathbf{A}||\mathbf{B}|$$

(b)

$$\begin{vmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} = |\mathbf{A}||\mathbf{B}| \quad \text{for} \quad |\mathbf{A}| \neq 0$$

*Hint:*

(a) $\begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} = \begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix} \begin{vmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix}$. Expanding the determinant $\begin{vmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix}$ by the first row (see Definition 2A.24) gives 1 times a determinant of the same form, with the order of $\mathbf{I}$ reduced by one. This procedure is repeated until $1 \times |\mathbf{B}|$ is obtained. Similarly, expanding the determinant $\begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix}$ by the last row gives $\begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix} = |\mathbf{A}|$.

(b) $\begin{vmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} = \begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} \begin{vmatrix} \mathbf{I} & \mathbf{A}^{-1}\mathbf{C} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix}$. But expanding the determinant $\begin{vmatrix} \mathbf{I} & \mathbf{A}^{-1}\mathbf{C} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix}$ by the last row gives $\begin{vmatrix} \mathbf{I} & \mathbf{A}^{-1}\mathbf{C} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix} = 1$. Now use the result in Part a.

**4.11.** Show that, if $\mathbf{A}$ is square,

$$|\mathbf{A}| = |\mathbf{A}_{22}||\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}| \quad \text{for } |\mathbf{A}_{22}| \neq 0$$
$$= |\mathbf{A}_{11}||\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}| \quad \text{for } |\mathbf{A}_{11}| \neq 0$$

*Hint:* Partition $\mathbf{A}$ and verify that

$$\begin{bmatrix} \mathbf{I} & -\mathbf{A}_{12}\mathbf{A}_{22}^{-1} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{0} \\ \mathbf{0}' & \mathbf{A}_{22} \end{bmatrix}$$

Take determinants on both sides of this equality. Use Exercise 4.10 for the first and third determinants on the left and for the determinant on the right. The second equality for $|\mathbf{A}|$ follows by considering

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0}' & \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \end{bmatrix}$$

**4.12.** Show that, for $\mathbf{A}$ symmetric,

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{I} \end{bmatrix} \begin{bmatrix} (\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21})^{-1} & \mathbf{0} \\ \mathbf{0}' & \mathbf{A}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{A}_{12}\mathbf{A}_{22}^{-1} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix}$$

Thus, $(\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21})^{-1}$ is the upper left-hand block of $\mathbf{A}^{-1}$.

*Hint:* Premultiply the expression in the hint to Exercise 4.11 by $\begin{bmatrix} \mathbf{I} & -\mathbf{A}_{12}\mathbf{A}_{22}^{-1} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix}^{-1}$ and postmultiply by $\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{I} \end{bmatrix}^{-1}$. Take inverses of the resulting expression.

**4.13.** Show the following if $\mathbf{X}$ is $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $|\boldsymbol{\Sigma}| \neq 0$.

(a) Check that $|\boldsymbol{\Sigma}| = |\boldsymbol{\Sigma}_{22}||\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}|$. (Note that $|\boldsymbol{\Sigma}|$ can be factored into the product of contributions from the marginal and conditional distributions.)

(b) Check that

$$(\mathbf{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}) = [\mathbf{x}_1 - \boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2)]'$$
$$\times (\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21})^{-1}[\mathbf{x}_1 - \boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2)]$$
$$+ (\mathbf{x}_2 - \boldsymbol{\mu}_2)'\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2)$$

(Thus, the joint density exponent can be written as the sum of two terms corresponding to contributions from the conditional and marginal distributions.)

(c) Given the results in Parts a and b, identify the marginal distribution of $\mathbf{X}_2$ and the conditional distribution of $\mathbf{X}_1 \mid \mathbf{X}_2 = \mathbf{x}_2$.

*Hint:*

(a) Apply Exercise 4.11.

(b) Note from Exercise 4.12 that we can write $(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})$ as

$$\begin{bmatrix} \mathbf{x}_1-\boldsymbol{\mu}_1 \\ \mathbf{x}_2-\boldsymbol{\mu}_2 \end{bmatrix}' \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21} & \mathbf{I} \end{bmatrix} \begin{bmatrix} (\boldsymbol{\Sigma}_{11}-\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21})^{-1} & \mathbf{0} \\ \mathbf{0}' & \boldsymbol{\Sigma}_{22}^{-1} \end{bmatrix} \times \begin{bmatrix} \mathbf{I} & -\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1-\boldsymbol{\mu}_1 \\ \mathbf{x}_2-\boldsymbol{\mu}_2 \end{bmatrix}$$

If we group the product so that

$$\begin{bmatrix} \mathbf{I} & -\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1-\boldsymbol{\mu}_1 \\ \mathbf{x}_2-\boldsymbol{\mu}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1-\boldsymbol{\mu}_1-\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2-\boldsymbol{\mu}_2) \\ \mathbf{x}_2-\boldsymbol{\mu}_2 \end{bmatrix}$$

the result follows.

**4.14.** If $\mathbf{X}$ is distributed as $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $|\boldsymbol{\Sigma}| \neq 0$, show that the joint density can be written as the product of marginal densities for

$$\underset{(q\times 1)}{\mathbf{X}_1} \quad \text{and} \quad \underset{((p-q)\times 1)}{\mathbf{X}_2} \quad \text{if} \quad \boldsymbol{\Sigma}_{12} = \underset{(q\times(p-q))}{\mathbf{0}}$$

*Hint:* Show by block multiplication that

$$\begin{bmatrix} \boldsymbol{\Sigma}_{11}^{-1} & \mathbf{0} \\ \mathbf{0}' & \boldsymbol{\Sigma}_{22}^{-1} \end{bmatrix} \text{is the inverse of } \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \mathbf{0} \\ \mathbf{0}' & \boldsymbol{\Sigma}_{22} \end{bmatrix}$$

Then write

$$\begin{aligned}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) &= [(\mathbf{x}_1-\boldsymbol{\mu}_1)', (\mathbf{x}_2-\boldsymbol{\mu}_2)'] \begin{bmatrix} \boldsymbol{\Sigma}_{11}^{-1} & \mathbf{0} \\ \mathbf{0}' & \boldsymbol{\Sigma}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1-\boldsymbol{\mu}_1 \\ \mathbf{x}_2-\boldsymbol{\mu}_2 \end{bmatrix} \\ &= (\mathbf{x}_1-\boldsymbol{\mu}_1)'\boldsymbol{\Sigma}_{11}^{-1}(\mathbf{x}_1-\boldsymbol{\mu}_1) + (\mathbf{x}_2-\boldsymbol{\mu}_2)'\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}_2-\boldsymbol{\mu}_2)\end{aligned}$$

Note that $|\boldsymbol{\Sigma}| = |\boldsymbol{\Sigma}_{11}||\boldsymbol{\Sigma}_{22}|$ from Exercise 4.10(a). Now factor the joint density.

**4.15.** Show that $\sum_{j=1}^{n}(\mathbf{x}_j-\bar{\mathbf{x}})(\bar{\mathbf{x}}-\boldsymbol{\mu})'$ and $\sum_{j=1}^{n}(\bar{\mathbf{x}}-\boldsymbol{\mu})(\mathbf{x}_j-\bar{\mathbf{x}})'$ are both $p \times p$ matrices of zeros. Here $\mathbf{x}_j' = [x_{j1}, x_{j2}, \ldots, x_{jp}]$, $j = 1, 2, \ldots, n$, and

$$\bar{\mathbf{x}} = \frac{1}{n}\sum_{j=1}^{n}\mathbf{x}_j$$

**4.16.** Let $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$, and $\mathbf{X}_4$ be independent $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ random vectors.

(a) Find the marginal distributions for each of the random vectors

$$\mathbf{V}_1 = \tfrac{1}{4}\mathbf{X}_1 - \tfrac{1}{4}\mathbf{X}_2 + \tfrac{1}{4}\mathbf{X}_3 - \tfrac{1}{4}\mathbf{X}_4$$

and

$$\mathbf{V}_2 = \tfrac{1}{4}\mathbf{X}_1 + \tfrac{1}{4}\mathbf{X}_2 - \tfrac{1}{4}\mathbf{X}_3 - \tfrac{1}{4}\mathbf{X}_4$$

(b) Find the joint density of the random vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ defined in (a).

**4.17.** Let $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4$, and $\mathbf{X}_5$ be independent and identically distributed random vectors with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Find the mean vector and covariance matrices for each of the two linear combinations of random vectors

$$\tfrac{1}{5}\mathbf{X}_1 + \tfrac{1}{5}\mathbf{X}_2 + \tfrac{1}{5}\mathbf{X}_3 + \tfrac{1}{5}\mathbf{X}_4 + \tfrac{1}{5}\mathbf{X}_5$$

and

$$\mathbf{X}_1 - \mathbf{X}_2 + \mathbf{X}_3 - \mathbf{X}_4 + \mathbf{X}_5$$

in terms of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. Also, obtain the covariance between the two linear combinations of random vectors.

**4.18.** Find the maximum likelihood estimates of the $2 \times 1$ mean vector $\boldsymbol{\mu}$ and the $2 \times 2$ covariance matrix $\boldsymbol{\Sigma}$ based on the random sample

$$\mathbf{X} = \begin{bmatrix} 3 & 6 \\ 4 & 4 \\ 5 & 7 \\ 4 & 7 \end{bmatrix}$$

from a bivariate normal population.

**4.19.** Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{20}$ be a random sample of size $n = 20$ from an $N_6(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ population. Specify each of the following completely.

(a) The distribution of $(\mathbf{X}_1 - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}_1 - \boldsymbol{\mu})$

(b) The distributions of $\overline{\mathbf{X}}$ and $\sqrt{n}(\overline{\mathbf{X}} - \boldsymbol{\mu})$

(c) The distribution of $(n - 1)\,\mathbf{S}$

**4.20.** For the random variables $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{20}$ in Exercise 4.19, specify the distribution of $\mathbf{B}(19\mathbf{S})\mathbf{B}'$ in each case.

(a) $\mathbf{B} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$

**4.21.** Let $\mathbf{X}_1, \ldots, \mathbf{X}_{60}$ be a random sample of size 60 from a four-variate normal distribution having mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$. Specify each of the following completely.

(a) The distribution of $\overline{\mathbf{X}}$

(b) The distribution of $(\mathbf{X}_1 - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}_1 - \boldsymbol{\mu})$

(c) The distribution of $n(\overline{\mathbf{X}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\overline{\mathbf{X}} - \boldsymbol{\mu})$

(d) The approximate distribution of $n(\overline{\mathbf{X}} - \boldsymbol{\mu})'\mathbf{S}^{-1}(\overline{\mathbf{X}} - \boldsymbol{\mu})$

**4.22.** Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{75}$ be a random sample from a population distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. What is the approximate distribution of each of the following?

(a) $\overline{\mathbf{X}}$

(b) $n(\overline{\mathbf{X}} - \boldsymbol{\mu})'\mathbf{S}^{-1}(\overline{\mathbf{X}} - \boldsymbol{\mu})$

**4.23.** Consider the annual rates of return (including dividends) on the Dow-Jones industrial average for the years 1996–2005. These data, multiplied by 100, are

$$-0.6 \quad 3.1 \quad 25.3 \quad -16.8 \quad -7.1 \quad -6.2 \quad 25.2 \quad 22.6 \quad 26.0.$$

Use these 10 observations to complete the following.

(a) Construct a $Q$–$Q$ plot. Do the data seem to be normally distributed? Explain.

(b) Carry out a test of normality based on the correlation coefficient $r_Q$. [See (4–31).] Let the significance level be $\alpha = .10$.

**4.24.** Exercise 1.4 contains data on three variables for the world's 10 largest companies as of April 2005. For the sales ($x_1$) and profits ($x_2$) data:

(a) Construct $Q$–$Q$ plots. Do these data appear to be normally distributed? Explain.

(b) Carry out a test of normality based on the correlation coefficient $r_Q$. [See (4-31).] Set the significance level at $\alpha = .10$. Do the results of these tests corroborate the results in Part a?

**4.25.** Refer to the data for the world's 10 largest companies in Exercise 1.4. Construct a chi-square plot using all *three* variables. The chi-square quantiles are

0.3518 0.7978 1.2125 1.6416 2.1095 2.6430 3.2831 4.1083 5.3170 7.8147

**4.26.** Exercise 1.2 gives the age $x_1$, measured in years, as well as the selling price $x_2$, measured in thousands of dollars, for $n = 10$ used cars. These data are reproduced as follows:

| $x_1$ | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 8 | 9 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|
| $x_2$ | 18.95 | 19.00 | 17.95 | 15.54 | 14.00 | 12.95 | 8.94 | 7.49 | 6.00 | 3.99 |

(a) Use the results of Exercise 1.2 to calculate the squared statistical distances $(\mathbf{x}_j - \bar{\mathbf{x}})'\mathbf{S}^{-1}(\mathbf{x}_j - \bar{\mathbf{x}})$, $j = 1, 2, \ldots, 10$, where $\mathbf{x}_j' = [x_{j1}, x_{j2}]$.

(b) Using the distances in Part a, determine the proportion of the observations falling within the estimated 50% probability contour of a bivariate normal distribution.

(c) Order the distances in Part a and construct a chi-square plot.

(d) Given the results in Parts b and c, are these data approximately bivariate normal? Explain.

**4.27.** Consider the radiation data (with door closed) in Example 4.10. Construct a $Q$–$Q$ plot for the natural logarithms of these data. [Note that the natural logarithm transformation corresponds to the value $\lambda = 0$ in (4-34).] Do the natural logarithms appear to be normally distributed? Compare your results with Figure 4.13. Does the choice $\lambda = \frac{1}{4}$ or $\lambda = 0$ make much difference in this case?

*The following exercises may require a computer.*

**4.28.** Consider the air-pollution data given in Table 1.5. Construct a $Q$–$Q$ plot for the solar radiation measurements and carry out a test for normality based on the correlation coefficient $r_Q$ [see (4-31)]. Let $\alpha = .05$ and use the entry corresponding to $n = 40$ in Table 4.2.

**4.29.** Given the air-pollution data in Table 1.5, examine the pairs $X_5 = NO_2$ and $X_6 = O_3$ for bivariate normality.

(a) Calculate statistical distances $(\mathbf{x}_j - \bar{\mathbf{x}})'\mathbf{S}^{-1}(\mathbf{x}_j - \bar{\mathbf{x}})$, $j = 1, 2, \ldots, 42$, where $\mathbf{x}_j' = [x_{j5}, x_{j6}]$.

(b) Determine the proportion of observations $\mathbf{x}_j' = [x_{j5}, x_{j6}]$, $j = 1, 2, \ldots, 42$, falling within the approximate 50% probability contour of a bivariate normal distribution.

(c) Construct a chi-square plot of the ordered distances in Part a.

**4.30.** Consider the used-car data in Exercise 4.26.

(a) Determine the power transformation $\hat{\lambda}_1$ that makes the $x_1$ values approximately normal. Construct a $Q$–$Q$ plot for the transformed data.

(b) Determine the power transformations $\hat{\lambda}_2$ that makes the $x_2$ values approximately normal. Construct a $Q$–$Q$ plot for the transformed data.

(c) Determine the power transformations $\hat{\boldsymbol{\lambda}}' = [\hat{\lambda}_1, \hat{\lambda}_2]$ that make the $[x_1, x_2]$ values jointly normal using (4-40). Compare the results with those obtained in Parts a and b.

**4.31.** Examine the marginal normality of the observations on variables $X_1, X_2, \ldots, X_5$ for the multiple-sclerosis data in Table 1.6. Treat the non-multiple-sclerosis and multiple-sclerosis groups separately. Use whatever methodology, including transformations, you feel is appropriate.

**4.32.** Examine the marginal normality of the observations on variables $X_1, X_2, \ldots, X_6$ for the radiotherapy data in Table 1.7. Use whatever methodology, including transformations, you feel is appropriate.

**4.33.** Examine the marginal and bivariate normality of the observations on variables $X_1, X_2, X_3$, and $X_4$ for the data in Table 4.3.

**4.34.** Examine the data on bone mineral content in Table 1.8 for marginal and bivariate normality.

**4.35.** Examine the data on paper-quality measurements in Table 1.2 for marginal and multivariate normality.

**4.36.** Examine the data on women's national track records in Table 1.9 for marginal and multivariate normality.

**4.37.** Refer to Exercise 1.18. Convert the women's track records in Table 1.9 to speeds measured in meters per second. Examine the data on speeds for marginal and multivariate normality.

**4.38.** Examine the data on bulls in Table 1.10 for marginal and multivariate normality. Consider only the variables YrHgt, FtFrBody, PrctFFB, BkFat, SaleHt, and SaleWt.

**4.39.** The data in Table 4.6 (see the psychological profile data: www.prenhall.com/statistics) consist of 130 observations generated by scores on a psychological test administered to Peruvian teenagers (ages 15, 16, and 17). For each of these teenagers the gender (male = 1, female = 2) and socioeconomic status (low = 1, medium = 2) were also recorded. The scores were accumulated into five subscale scores labeled *independence* (indep), *support* (supp), *benevolence* (benev), *conformity* (conform), and *leadership* (leader).

**Table 4.6** Psychological Profile Data

| Indep | Supp | Benev | Conform | Leader | Gender | Socio |
|---|---|---|---|---|---|---|
| 27 | 13 | 14 | 20 | 11 | 2 | 1 |
| 12 | 13 | 24 | 25 | 6 | 2 | 1 |
| 14 | 20 | 15 | 16 | 7 | 2 | 1 |
| 18 | 20 | 17 | 12 | 6 | 2 | 1 |
| 9 | 22 | 22 | 21 | 6 | 2 | 1 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 10 | 11 | 26 | 17 | 10 | 1 | 2 |
| 14 | 12 | 14 | 11 | 29 | 1 | 2 |
| 19 | 11 | 23 | 18 | 13 | 2 | 2 |
| 27 | 19 | 22 | 7 | 9 | 2 | 2 |
| 10 | 17 | 22 | 22 | 8 | 2 | 2 |

Source: Data courtesy of C. Soto.

(a) Examine each of the variables independence, support, benevolence, conformity and leadership for marginal normality.

(b) Using all five variables, check for multivariate normality.

(c) Refer to part (a). For those variables that are nonnormal, determine the transformation that makes them more nearly normal.

**4.40.** Consider the data on national parks in Exercise 1.27.

(a) Comment on any possible outliers in a scatter plot of the original variables.

(b) Determine the power transformation $\hat{\lambda}_1$ the makes the $x_1$ values approximately normal. Construct a $Q-Q$ plot of the transformed observations.

(c) Determine the power transformation $\hat{\lambda}_2$ the makes the $x_2$ values approximately normal. Construct a $Q-Q$ plot of the transformed observations.

(d) Determine the power transformation for approximate bivariate normality using (4-40).

**4.41.** Consider the data on snow removal in Exercise 3.20.

(a) Comment on any possible outliers in a scatter plot of the original variables.

(b) Determine the power transformation $\hat{\lambda}_1$ the makes the $x_1$ values approximately normal. Construct a $Q-Q$ plot of the transformed observations.

(c) Determine the power transformation $\hat{\lambda}_2$ the makes the $x_2$ values approximately normal. Construct a $Q-Q$ plot of the transformed observations.

(d) Determine the power transformation for approximate bivariate normality using (4-40).

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